Viable Capture Basin for Studying Differential and Hybrid Games: Application to Finance

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Abstract
Viability theory can be applied for determining viable capture basin for control problem in presence of uncertainty. We first recall the concepts of viability theory which allow to develop numerical methods for computing viable capture basin for control problems and guaranteed control problems. Recent developments of option pricing in the framework of dynamical games with constraints lead to the formulation of guaranteed valuation in terms of guaranteed viable-capture basin of a dynamical game. As an application we show how the viability/capturability algorithm evaluates and manages portfolios. Regarding viability/capturability issues, stochastic control is a particular use of tychastic control. We replace the standard translation of uncertainty by stochastic control problem by tychastic ones and the concept of stochastic viability by the one of guaranteed viability kernel. Considering the Cox-Rubinstein model, we extend algorithms for hedging portfolios in the presence of transaction costs and dividends using recent developments on hybrid calculus.

Keywords: Viability, Capturability, Tychastic control, Dynamical Games, Hedging portfolio.

1. Introduction
In this paper we present some applications of Viability Theory and Set Valued Numerical Analysis to the problem of hedging portfolio with transaction costs. We first recall and illustrate the concept of Viable Capture Basin in the framework of control problems and the concept of Guaranteed Viable Capture Basin in the framework of differential Games. Then we show how these notions of “capturability” of a target and of viability of a system, under constraints can be applied for evaluating portfolios in the general case and how the Capture Basin Algorithm can be fruitfully used to determine numerically the rules for managing a portfolio (“Pujal (2000)”, “Pujal & Saint-Pierre (2001)”). Our scope is to emphasize the articulation between Viability, Games theory and Mathematical Finance, following the ideas developed in “Bernhard (2000), (2002)” and “Pujal (2000)” that appeared simultaneously and independently at the end of the year 2000. They consider the evolution of the prices.
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governed by

\[ \forall i = 1, \ldots, n, \quad x_i'(t) = x_i(t)\rho_i(x(t), v(t)) \text{ where } v(t) \in Q(x(t)) \]

where \( v(t) \) is regarded as a typo, a perturbation, a disturbance. We consider the problem of evaluation of portfolio in the presence of transaction cost. This has been studied in “Aubin, Pujal & Saint-Pierre (2001)” and “Bernhard (2002)”. We present here new algorithms for evaluating portfolio and finding hedging strategies and we provide some numerical results. We will not give any proof of existence of solutions or convergence of algorithm that can be found in referenced papers.

2. Viability Kernels & Viable Capture Basins

Let us consider the differential control system

\[ x' = f(x, u), \text{ where } u \in U(x) \]  \hspace{1cm} (2.1)

The viable capture basin of a closed target \( C \) viable in a closed set \( K \) is the set of elements \( x \in K \) such that there exists a continuous feedback \( \bar{u}(x) \in U(x) \) and \( t^* \in \mathbb{R}_+ \) such that the solution \( x(\cdot) \) to \( x' = f(x, \bar{u}(x)) \) exists and satisfies

\[
\begin{align*}
& i) \quad x(t) \in K, \quad \forall t \in [0, t^*] \\
& ii) \quad x(t^*) \in C
\end{align*}
\]

This set is denoted \( \text{Capt}_F(K, C) \). Existence and properties of this set can be found in “Aubin (1991)”. In general there is no way to describe it analytically. However it can be numerically approximated thanks to the Viability Kernel Algorithm.

Let \( F_C \) be the set-valued map which coincides with \( F \) outside \( C \), equals to 0 on \( \text{Interior}(C) \) and equals to \( \text{Conv}(\{0\} \cup F(x)) \) on \( \partial C \). If \( K \) is a repeller under \( F \), then \( \text{Capt}_F(K, C) = \text{Viable}_F(K) \). Otherwise, one can characterize the viable capture basin as the domain of the Minimal Time function defined by

\[ \vartheta^K_C(x) := \inf_{x(\cdot) \in F(x)} \{ \tau | x(\tau) \in C, \quad x(t) \in K, \forall t \leq \tau \} \]

which satisfies the relation \( \text{Epigraph}(\vartheta^K_C) = \text{Viable}_F \times (-1)(K \times \mathbb{R}_+) \) so that

\[ \text{Capt}_F(K, C) = \text{Proj}_X \{ \text{Viable}_F \times (-1)(K \times \mathbb{R}_+) \} \]

To illustrate this we consider the following examples:

**Example 1.** [Minimal time for the Labyrinth Problem]

Let us consider the dynamical system

\[ (x'(t), y'(t), z'(t)) = f(x(t), y(t), z(t), u, v) := (u, v, -1), \text{ with } u^2 + v^2 \leq 1 \]

with the target \( C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1 \} \) and the constraints \( K = \{[-1.2, 1.2] \times [-1.2, 1.2] \} \setminus M \) where \( M \) are obstacles.

The Capture Basin of \( C \) under \( K \) for \( f \) is the domain of the minimal time function - which coincides with \( K \) - and the epigraph of the minimal time function.
is the Viability Kernel of $K \times R^+$ for $f$.
This function is the smallest lower semi-continuous super solution of the Hamilton Jacobi-Belmann equation:
\[
\min_{(u,v) \in B} \left( u \frac{\partial \vartheta}{\partial x} + v \frac{\partial \vartheta}{\partial y} \right) \leq -1
\]
on $K \setminus C$

\subsection{Viable Guaranteed and Conditional Capture Basins}

Let us now consider the two-player differential game characterized by the differential system
\[
x' = f(x, u, v), \text{ where } u \in U(x) \text{ and } v \in V(x)
\]
and the two-players discrete game described by the recursive system
\[
x^{n+1} = g(x^n, u^n, v^n)
\]
with a constraint set $K$. Two kinds of games with constraints involve targets, the first one is a capture problem with an open target and the second one is a minimal time problem with a close target (see "Cardaliaguet, Quincampoix & Saint-Pierre (1995), (2001)").

We extend the concept of Viable Capture Basin up to differential games and define

- the **guaranteed viable-capture basin** which is the set of $x \in K$ such that there exists a continuous selection $\widetilde{u}(x) \in U(x)$ such that $\forall v(\cdot) \in V(x(\cdot)), \exists t^* \in R_+$ such that the solution $x(\cdot)$ to
\[
x'(t) = f(x(t), \widetilde{u}(x(t)), v(t))
\]
exists and satisfies $x(t) \in K, \forall t \in [0, t^*]$ and $x(t^*) \in C$.

- the **conditional viable-capture basin** which is the set of $x \in K$ such that for any continuous selection $\widetilde{v}(x) \in V(x), \exists u(\cdot) \in U(x(\cdot))$ such that $\exists t^* \in R_+$ such that the
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solution $x(\cdot)$ to

$$x'(t) = f(x(t), u(t), v(x(t)))$$

exists and satisfies $x(t) \in K, \forall t \in [0, t^*]$ and $x(t^*) \in C$.

These domains can be characterized by geometric condition:

- The Viability Kernel: $\text{Viab}_F(K)$ is the largest closed subset $D \subseteq K$ such that $\forall x \in D, \exists u \in U(x) : f(x, u) \in T_D(x)$
- The Discrete Viability Kernel: $\overline{\text{Viab}}_G(K)$ is the largest closed subset $D \subseteq K$ such that $\forall x \in D, \exists u \in U(x) : g(x, u) \in D$
- The Discrete Conditional Viability Kernel: $\overline{\text{Cond}}_F(K)$ is the largest closed subset $D \subseteq K$ such that $\forall x \in D, \exists u \in U(x), \exists v \in V(x) : g(x, u, v) \in D$
- The Discrete Guaranteed Viability Kernel: $\overline{\text{Guar}}_F(K)$ is the largest closed subset $D \subseteq K$ such that $\forall x \in D, \exists u \in U(x), \exists v \in V(x) : g(x, u, v) \in D$

2.2. Discrete Constrained Games Algorithms

Thanks to these properties we can design algorithms for approximate viable guaranteed or conditional capture basin. For this task let us consider a discrete game given by

$$x_{n+1} = g(x^n, u^n, v^n) = x^n + \rho f_{\rho}(x^n, u^n, v^n)$$

where $\rho$ denotes the time step and $f_{\rho} = f + \varphi(\rho)B$.

We can approach Discrete Guaranteed Viability Kernel and Discrete Conditional Viability Kernel by extending the Viability Kernel Algorithm and constructing decreasing sequences of closed sets defined recursively as follows (“Cardaliaguet, Quincampoix & Saint-Pierre (1999))

- **Discrete Guaranteed Viability Kernel:**
  \[ K_{\rho, g}^{0} = K, \quad K_{\rho, g}^{n+1} = \{ x \in K_{\rho, g}^{n} | \exists u \in U(x), \forall v \in V(x), g_{\rho}(x, u, v) \in K_{\rho, g}^{n} \} \]
  and $\text{Guar}_{\rho, g}(K) = K_{\rho, g}^{\infty}$

- **Discrete Conditional Viability Kernel:**
  \[ K_{\rho, c}^{0} = K, \quad K_{\rho, c}^{n+1} = \{ x \in K_{\rho, c}^{n} | \forall v \in V(x), \exists u \in U(x), g_{\rho}(x, u, v) \in K_{\rho, c}^{n} \} \]
  and $\text{Cond}_{\rho, c}(K) = K_{\rho, c}^{\infty}$

3. Application: Hedging Portfolio without or with Transaction Costs

Let us now consider the problem of evaluating an hedging portfolio in the framework of dynamical games with constraints where the epigraph of the claim function will play the role of the target.
We discuss here only the discrete model of evaluation but we will comment some questions arising when considering continuous models with both uncertainty and transaction costs.

Let be $T$ the time left until maturity. Let $S_0$, $S_1$ and $W$ denote the bond, the values of the risky asset and of the hedging portfolio. Let $S = (S_0, S_1)$. The variable $x$ corresponds to $(T, S, W)$ and $p$ is the control. Parameters $\gamma_0(S_0)$ and $\gamma_1(S_1, v)$ measure the rates of price evolution, $v$ describes stochastic uncertainty. $T$ is the exercise time and $K$ is the striking price*.

Let us consider a given time-independent function $u : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$, called the contingent claim$^1$.

The general model can be interpreted, in the discrete formulation, as a problem of finding initial conditions $(T, S_1, W)$ such that there exists a feedback $S \mapsto \pi(S) \in P$ such that, whatever the perturbation $v^n \in [v_m, v_M]$ is, the hedging portfolio satisfies a viability condition

$$\forall n \leq N, \quad W^n \geq b(T - t^n, S^n) \quad \iff \quad (T - t^n, S^n, W^n) \in Epi(b) \quad (3.2)$$

where $b$ is a given constraint function defined on $\mathbb{R} \times \mathbb{R}^+$, with values in $\mathbb{R} \cup +\infty$, and a capturability condition:

1. **European Option**:
   $$W^N = p_0^N S_0^N + p_1^N S_1^N \geq u(S^N) \quad \iff \quad (S^N, W^N) \in Epi(u) \quad (3.3)$$

2. **American Option**:
   $$W^n = p_0^n S_0^n + p_1^n S_1^n \geq u(S^n), \forall n \leq N, \quad \iff \quad (S^n, W^n) \in Epi(u) \quad (3.4)$$

3. **First Time Options**: the option is exercised at the first time step $n^* \leq N$ when
   $$W^{n^*} = p_0^{n^*} S_0^{n^*} + p_1^{n^*} S_1^{n^*} \geq u(S^{n^*}) \quad \iff \quad (S^{n^*}, W^{n^*}) \in Epi(u) \quad (3.5)$$

In order to treat the three rules of the three games as particular cases of a more general framework, we introduce a nonnegative extended functions $c$ (objective function) satisfying

$$\forall (t, S) \in \mathbb{R} \times \mathbb{R}_+, \quad 0 \leq b(t, S) \leq c(t, S) \leq +\infty$$

and

$$\forall t < 0, \quad b(t, S) = c(t, S) = +\infty$$

* $K$ is the usual notation in Finance, without confusion with the previous notation for the constraint state.

$^1$ For example, one can choose $u(S) = (S_1 - K)^+$ but any lower semicontinuous map can be considered.
By associating with the initial function $u$ adequate pairs $(b,c)$ of extended functions, we shall replace the requirements (3.3,3.4,3.5) by the requirement

\[
\begin{align*}
  (i) \quad & \forall n \leq N, \quad (T - t^n, S^n, W^n) \in \text{Epi}(b) \\
  \text{(dynamical constraint)} \\
  (ii) \quad & (T - t^n, S^n, W^n) \in \text{Epi}(c) \\
  \text{(final objective)}
\end{align*}
\] (3.6)

Allowing the functions to take infinite values (i.e., to be extended), allows us to acclimate many examples.

So the three rules associated with a same function $u : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ can be written in the form (3.6) by adequate choices of pairs $(b,c)$ of functions associated with $u$. Indeed, denoting by $0$ the function defined by

\[
0(t,S) = \begin{cases} 
0 & \text{if } t \geq 0, \\
+\infty & \text{if not}
\end{cases}
\]

and by $u_\infty$ the function defined by

\[
u_\infty(t,S) := \begin{cases} 
u(S) & \text{if } t = 0 \\
+\infty & \text{if not}
\end{cases}
\] (3.7)

we can recover the three rules of the game

1. We take $b(t,S) := 0$ and $c(t,S) = u_\infty(t,S)$, we obtain the rule for the European option,

2. We take $b(t,S) := u(S)$ and $b(t,S) := u_\infty(t,S)$, we obtain the rule for the American option,

3. We take $b(t,S) := 0$ and $c(t,S) = u(S)$, we obtain the rule for the first time option.

### 3.1. Portfolio Evaluation without transaction cost

Without transaction cost, since $W^n = \sum_{i=0}^{n+1} p^n_i S^n_i$, we introduce the self-financing assumption $\sum_{i=0}^{n+1} (p^n_{i+1} - p^n_i) S^n_{i+1} = 0$ and the discrete evolution of the finance system becomes

\[
\begin{align*}
  T^{n+1} &= T^n - \rho \\
  S^n_{i+1} &= S^n_i (1 + \gamma_{p^n}(S^n_i, v^n)), \quad i = 0, 1 \\
  W^n_{i+1} &= W^n_i (1 + \gamma_{p^n}(S^n_i, v^n)) + p^n_i S^n_{i+1} (\gamma_{p^n}(S^n_{i+1}, v^n) - \gamma_{p^n}(S^n_i))
\end{align*}
\]

where quantities $p_0$ and $p_1$ play the role of controls, $p_0 \leq 0, p_1 \in [0,1]$ and $N$ denotes the number of intervals of time at which end the hedging portfolio is reevaluated which length is $\rho$.

The uncertainty over the period of time $[t^n, t^n + \rho]$ is represented by a tyche $v \in Q_{p^n}(S^n)$ (which may depends on $n$).
The choice of $Q_\rho(S)$ determines the degree of uncertainty distributed on each small time intervals $\rho = T/N$.

If $\gamma_1(S_1, v) = \rho \gamma_1(S_1) + v$ with $v \in Q_\rho(S) = [v_m, v_M] = [\epsilon^{-\sigma^2\rho} - 1, e^{\sigma^2\rho} - 1]$, 
- the "stochastic" or "contingent" uncertainty corresponds to $\lambda = 1$. In this case we get the guaranteed or conditional contingent evaluation corresponding to situations where the uncertainty is not stochastic.
- the Cox & Rubinstein model corresponds to $\lambda = 1/2$ that is the up and down volatility terms in the time refined binomial model\(^1\).

The target is $C = (T, Epi(u))$ and the constraint set is $K = Epi(b)$.

The Discrete Guaranteed and Conditional Capture Basin Algorithms that we briefly recalled in the previous section amounts to define sequences of functions $V_{\rho, g}^{n+1}$ and $V_{\rho, c}^{n+1}$ which epigraphs are precisely sets $K_{\rho, g}^n$ and $K_{\rho, c}^n$. This leads to the following construction. Let us start the sequences with

$$V_{\rho, g}^0(t, S_1) = V_{\rho, c}^0(t, S_1) = \begin{cases} +\infty, & \text{if } t < 0 \\ u(S_1), & \text{if } 0 \leq t < \rho \\ b(t, S_1), & \text{if } \rho \leq t \leq T \end{cases}$$

\(^1\)Let us recall that when $h_N$ goes to 0 the evaluation function of European option converges to the Black & Scholes value that is to say to stochastic uncertainty (“Bernhard (2002)”). In this case we can approximate the value given by the Black & Scholes formula which in some sense calibrates the Algorithm.
We set $V_{\rho,g}(t, s) = V^n_{\rho,g}(t, S_1), \; \forall t \in [(n-1)\rho, n\rho]$.

The approximation of the conditional valuation function is given by

$$ V^{n+1}_{\rho,c}(t, S_1) := \max \left( V^n_{\rho,c}(t, S_1), \sup_{v \in [v_m,v_M]} \inf_{p \in [0,1]} \inf_{\gamma, \varphi, \varphi'} \left( \frac{V^n_{\rho,c}(t - \rho, S_1(1 + \gamma \rho_1(S_1, v) + \beta)) - p_1 S_1(\gamma \rho_1(S_1, v) - \gamma_0)}{1 + \gamma_0} \right) \right) $$

We set $V_{\rho,c}(t, s) = V^n_{\rho,c}(t, S_1), \; \forall t \in [(n-1)\rho, n\rho]$.

**Example 2.** [Guaranteed Evaluation of European Call with Cox-Rubinstein type of uncertainty]

The following figures provide numerical results obtained with Algorithm O.

Figure 2 left shows the target Epigraph($u$) at $t = 0$ (in the plane $t = 0$), the graph of the evaluation function $V_{\rho,c}(t, s)$ and the value function of the call at maturity (in the plane $t = T = 1$).

Figure 2 right shows the same elements projected on the plane $(S, W)$.

Figure 3 left shows the optimal guaranteed policy for hedging portfolio by a color scaling superposed on the graph of the evaluation function.

Table 1 gives values of a call using the Capture Basin Method when the riskless asset growth rate is $\gamma_0 = 5\%$, and the maturity price is $K = 100$.

<table>
<thead>
<tr>
<th>Number of periods</th>
<th>value of $S_1$</th>
<th>Value of the Call</th>
<th>$p_0$</th>
<th>$p_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 512$</td>
<td>80</td>
<td>4.53</td>
<td>-22.10</td>
<td>0.3332</td>
</tr>
<tr>
<td>$N = 512$</td>
<td>90</td>
<td>8.61</td>
<td>-34.94</td>
<td>0.4844</td>
</tr>
<tr>
<td>$N = 512$</td>
<td>100</td>
<td>14.24</td>
<td>-48.17</td>
<td>0.6240</td>
</tr>
<tr>
<td>$N = 512$</td>
<td>110</td>
<td>21.04</td>
<td>-59.94</td>
<td>0.7363</td>
</tr>
<tr>
<td>$N = 512$</td>
<td>120</td>
<td>28.84</td>
<td>-69.63</td>
<td>0.8207</td>
</tr>
</tbody>
</table>

Table 1.

Fig. 3. On the left: optimal part $\pi_0(T,x)$ of non risky asset. On the right, contingent (stochastic) conditional evaluation of a non standard European call with varying rate and uncertainty.

**Example 3.** [Guaranteed Evaluation of European Call with varying rate and stochastic uncertainty]
In this example the claim function is given by
\[ u(S) = \min \left( \frac{(S-K)^+}{K^{sup}}, K^{sup} \right). \]

with \( \rho_1(S) = \frac{S}{1000} \) and \( \sigma_\rho(t) = 0.3\rho \frac{1}{0.01+\tau^2} \).

Figure 3 right shows the graphical result obtained with Algorithm 0 for computing the evaluation function of such a European call with varying rate and varying uncertainty.

3.2. Portfolio Evaluation with transaction cost

We can extend the Capture Basin Algorithm (CBA) designed for evaluating options for self-financed portfolio without transaction costs (“Pujal & Saint-Pierre (2001)”)) to the case when in the presence of proportional and/or fix transaction costs.

Functions \( u, b \) and \( c \) are now depending on variables \((t, S, P)\) and variable \( x \) corresponds to \((T, S, P, W)\). Parameter \( \rho_1 \) is considered as a variable which derivative denoted \( u \) now play the role of the control. Let \( \alpha_1(\rho_1) \) represent the rate of the transaction cost. Let \( \beta_1 \) represent the cost that one have to pay as soon as the composition of portfolio is altered: \( \beta_1(u) \) is equal to \( \beta_1 \) if \( u \neq 0 \) and 0 if \( u = 0 \). Even if the theoretical study of this problem is out of the scope of this paper, we can easily extend the algorithm up to the case of fix transaction costs as long as we keep \( N \) bounded.

The self-financing assumption becomes
\[ \sum_{i=0}^{1} p_i'(t)x_i(t) + \alpha_1 |p_i'(t)|S_i(t) + \beta_1(p_i(t)) = 0 \]

and the discrete system describing the evolution of the finance items reads:

\[
\begin{align*}
T^{n+1} &= T^n - \rho \\
S_i^{n+1} &= S_i^n (1 + \gamma_{\rho_1}(S_i^n, v^n)), \quad i = 0, 1 \\
p_1^{n+1} &= p_1^n + \rho u^n \\
W^{n+1} &= W^n (1 + \gamma_{\rho_0}(S_0^n)) + p_1^n S_1^n (\gamma_{\rho_1}(S_1^n, v^n)) \\
&\quad - \gamma_{\rho_0}(S_0^n) - \rho |u^n| \alpha_1 S_1^n - \beta_1(p_1^n(t)) \\
\end{align*}
\]

(3.8)

where \( u^n \in U(p^n) = [-\frac{1}{2}p_1^n, \frac{1}{2}(1 - p_1^n)] \) and \( v^n \in Q_\rho(S^n) \)

We look for the subset of initial conditions \((S_1^0, p_1^0, W_0)\) for which there exists a feedback \( S \to u(S) \in U \) such that, whatever the perturbation \( v^n \in Q(S^n, \rho) \) is, the successive values of the portfolio satisfy the viability-capturability conditions.

The Discrete Guaranteed Capture Basin Algorithm with Transition Costs leads to the construction of a sequence of sets \( K_\rho^n \) recursively defined by

\[
K_\rho^0 = \{ (\tau, S_1, p_1, y) \in [0, T] \times R^+ \times [0, 1] \times R^+ \text{ such that} \\
(T - \tau, S_1, p_1, y) \in \text{Epi}(c), \text{ if } \tau \leq \rho \\
(T - \tau, S_1, p_1, y) \in \text{Epi}(b), \text{ if } \tau \in [\rho, T] \}
\]

(3.9)

and, for \( n \geq 0 \),
The function \( L^{\text{Lipschitz}} \) denotes the ball contained in \( g \) where

\[
g(\tau, S_1, p_1, y, u, v) + \rho \phi \in K_{\rho}^n
\]

\[K_{\rho}^{n+1} = \{ (\tau, S_1, p_1, y) \in K_{\rho}^n \text{ such that } \exists \phi \in \phi(\rho)B, \exists u \in U(p), \forall v \in Q_\rho(S_1), \]

\[g(\tau, S_1, p_1, y, u, v) + \rho \phi \in K_{\rho}^n\]

where \( g(\tau, S_1, p_1, y, u, v) \) denotes the right hand side of relation (3.8) and \( \phi(\rho)B \) denotes the ball contained in \( R^3 \) of radius \( \phi(\rho) \) which decreases to 0 when \( \rho \to 0 \). The function \( \phi(\rho) \) depends on the regularity parameters of the map \( g \); in the Lipschitz case, \( \phi(\rho) = \frac{1}{2}M_t \phi \) (“Saint-Pierre (1994)”). This corrective function \( \phi(\rho) \) needs to be introduced when we are looking for limit solutions when \( N \to +\infty \).

If we only consider discrete evolution of portfolio with fixed \( N \) steps, we choose \( \phi(\rho) \equiv 0 \) as we have done in the next numerical examples.

### ALGORITHM I (with proportional transaction costs)

We suppose here that there is no fix transaction cost: \( \beta_1(p_1) = 0 \). With the sequence \( K_{\rho}^n \) we associate the sequence of maps \( (\tau, S_1, p_1) \to W_\rho^n(\tau, S_1, p_1) \) defined by

\[W_\rho^n(\tau, S_1, p_1) := \inf \{ y \text{ such that } (\tau, S_1, p_1, y) \in K_{\rho}^n \}\]

and from relation (3.10) it comes

\[W_\rho^{n+1}(\tau, S_1, p_1) := \max \left( W_\rho^n(\tau, S_1, p_1), \inf_{\phi \in \phi(\rho)B} \inf_{u \in U(S_1, p_1)} \sup_{v \in Q(S_1, \rho)} \frac{W_\rho^n(\tau - \rho, S_1(1 + \rho\gamma_1(S_1, v)) + \rho\phi, p_1 + \rho u - \rho(p_1 + \rho u)S_1(\gamma_1(S_1, v) - \gamma_0) + \rho\alpha_1|u|S_1}{1 + \rho\gamma_0} \right)\]

with

\[W_\rho^0(t, S_1, p_1) = \begin{cases} +\infty & \text{if } t < 0 \\ c(S_1, p_1) & \text{if } t \leq \rho \\ b(S_1, p_1) & \text{if } t > \rho \end{cases}\]

(3.12)

and the discrete guaranteed evaluation function is given by

\[W_\rho(t, S_1, p_1) = W_\rho^n(t, S_1, p_1), \forall t \in \left\{ (n - 1)\rho, n\rho \right\}\]

which epigraph is precisely the viable capture basin associated with (3.8) and \( W_\rho(T, S_1, p_1) \) is the minimal value of the hedging portfolio for the exercise time \( T \) is the value of the risky asset \( S_1 \) and if the portfolio contains \( p_1 \).

#### Example 4. [European call with transaction costs under tychastic uncertainty]

In the next example we choose \( T = 1, K = 100, \sigma := 0.3, \gamma_0 = 0 \). Uncertainty is tychastic with \( \gamma_1(v) = 0.1 + v, v \in [-\sigma, +\sigma] \) and the transaction cost rate \( \alpha_1 = 0.01 \).

On figure 5, evaluation functions \( (x, p) \mapsto V_\rho(T, S_1, p) \) are computed for different values of \( T \) with \( \alpha_1 = 1\% \), and on figure 6, they are superposed.

#### 3.3. The underlying Hamilton-Jacobi-Isaacs Equation

Without fix transaction cost we can prove that, when \( N \to \infty \), the approximated valuation function recovers the solution of the Hamilton-Jacobi-Isaacs inequality we have obtained above. From the definition of \( W_\rho \) given by (3.11), we deduce
Fig. 4. ALGORITHM I : Guaranteed evaluation function with Cox-Rubinstein uncertainty of European call with transaction costs.

\[ W_\rho(\tau + \rho, S_1, p_1) := \max \left( W_\rho(\tau, S_1, p_1), \inf_{\phi \in [\varphi(\rho), \varphi(\rho)]} \sup_{v \in Q(S_1, N)} \inf_{u \in U(\rho N)} W_\rho(\tau, S_1(1 + \rho(\gamma_1(S_1, v) + \phi)), p_1 + \rho u) - \rho(p_1 + \rho u)S_1(\gamma_1(S_1, v) - \gamma_0) + \rho|u|\alpha_1S_1 \right) \]

So that

\[ W_\rho(\tau + \rho, S_1, p_1) - W_\rho(\tau, S_1, p_1) = \max \left(0, \inf_{\phi \in [\varphi(\rho), \varphi(\rho)]} \sup_{v \in Q(S_1, N)} \inf_{u \in U(\rho N)} W_\rho(\tau, S_1(1 + \rho(\gamma_1(S_1, v) + \phi)), p_1 + \rho u) - W_\rho(\tau, S_1, p_1), \rho(1 + \rho \gamma_0) \right) \]

Assume that \( \gamma_1(S_1, v) = \gamma_1 + v \), it comes

\[ \Delta \gamma_1 W_\rho(\tau, S_1, p_1) = \min_{\phi \in [\varphi(\rho), \varphi(\rho)]} \sup_{v \in Q(S_1, N)} \min_{\phi \in [\varphi(\rho), \varphi(\rho)]} \sup_{v \in Q(S_1, N)} (\Delta S_1, W_\rho(\tau, S_1, p_1))(\gamma_1 + v)S_1 + \phi) + \Delta \rho W_\rho(\tau, S_1, p_1)u \]

\[ + (-\gamma_0 W_\rho(\tau, S_1) - (p_1 + \rho u)S_1(\gamma_1 + v - \gamma_0) + |u|\alpha_1S_1) \]

\[ \geq 0 + o(\rho) \]

The infimum with respect to \( \phi \) expresses the property that we approximate the Epi-
Fig. 5. Evaluation functions $V(x, p) \rightarrow V_p(T, S_1, p)$ are computed for different values of maturity $T$ with $\alpha_1 = 10\%$.

Fig. 6. Superposed evaluation functions computed for different values of maturity.
where \( v_x \) and \( v_p \) are the derivatives of \( S_1 \) and \( p_1 \).

Then, when \( \rho \to 0 \), \( W_\rho(\tau, S_1, p_1) \) converges to the lowest solution to the Hamilton-Jacobi-Isaacs variational inequality (“Frankowska (1993)”), as formulated in the previous sections

\[
\frac{\partial}{\partial \tau} W(\tau, S_1) - \inf_{u \in U(p_1)} \sup_{v \in Q(u)} D(\tau, S_1, p_1, f(\tau, S_1, p_1, u, v) + I(S_1, p_1, u, v) + m(S_1, p_1, v) W(\tau, S_1, p_1) \geq 0
\]

(3.16)

where \( f(\tau, S_1, p_1, u, v) \) describes the evolution of \( t \), \( S_1 \) and \( p_1 \),
\( m(S_1, p_1, v) = -\gamma_0 \) and \( I(S_1, p_1, u, v) = -(p_1 + \rho u)S_1(\gamma_1 + v - \gamma_0) + |u|\alpha_1S_1 \).

### 3.4. The One Period Case or the First Step Evaluation in the \( N \) Periods case.

If \( T = 1 \) and \( N = 1 \) for the one period case or \( N \) fixed for the \( N \) period case, then the following formula gives the evaluation function for the hedging portfolio for an one step procedure with \( \rho = \frac{T}{N} \):

\[
W_1(T, S_1, p_1) := \max\{b(S_1), \inf_{u \in [-1, 1]} \sup_{v \in [\sigma, 0]} (S_1 + \rho\gamma_1(S_1, v) - \gamma^+ - \rho(p_1 + \rho u)S_1(\gamma_1(S_1, v) - \gamma_0) + \rho \alpha_1 |u|S_1 + \beta_1(u)) \}
\]

(3.17)

Choosing \( p_1 = 0 \), assuming \( b = 0 \); \( \gamma_0 = 0 \), setting \( \gamma^+_1 = \gamma_1(S_1, \sigma) \) and \( \gamma^-_1 = \gamma_1(S_1, -\sigma) \) with \( \sigma = 0.3 \) the previous expression reads:

\[
W_1(T, S_1) := \min \{ (S_1 + \rho\gamma^+_1)(\gamma^-_1 - K^+), \max \{ (S_1 + \rho\gamma^+_1)(\gamma^-_1 - K^+ - \rho S_1(\gamma^+_1 + \alpha_1), S_1(1 + \rho\gamma^-_1)(\gamma^-_1 - K^+ - \rho S_1(\gamma^-_1 + \alpha_1) + \beta_1) \}
\]

(3.18)

When the riskless asset rate is set to \( \gamma_0 = 5\% \), the exercise price \( K = 100 \), \( T = 1 \) and \( N = 1 \), \( \gamma_1(v) = 0.1 + v \), \( v \in [-0.3, 0.3] \) and the fix cost \( \beta_1 = 0.00 \) and \( \beta_1 = 0.01 \), we get the following numerical values:

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<th>0.00</th>
<th>0.01</th>
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<th>0.03</th>
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<td>1.17-1.17</td>
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<td>1.47-1.49</td>
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<td>2.62-2.63</td>
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Fig. 8. Option Value in the one period case

### 3.5. The \( N \) Periods Case
Fig. 7. Value function after one step: comparison results for $N = 1$ or $N = 1000$ and tychastic ($\gamma_1 = \pm \sigma \frac{T}{N}$) or Cox & Rubinstein ($\gamma_1 = e^{\pm \sigma \sqrt{\frac{T}{N}}} - 1$) uncertainty.
Curves 1: $(S_1(1 + \gamma_1^+) - K)^+ - S_1(\gamma_1^+ + \alpha_1) + \beta_1$ ; 2: $(S_1(1 + \gamma_1^-) - K)^+ - S_1(\gamma_1^- + \alpha_1) + \beta_1$ ;
3: $(S_1(1 + \gamma_1^+) - K)^+$, 4: $W_1(T, S_1) = \min(Curve 3, \max(Curve 1, Curve 2))$. 
We are now looking for the behavior of the numerical approximation of the value function for a small number of steps $N \leq 20$ first when there is a proportional cost and second when there is both a proportional and a fix cost.

![Evaluation Function with transaction costs: CBATC algorithm with $\alpha_1 = 1\%$ and $\beta_1 = 0$, Cox & Rubinstein uncertainty](image)

Let us first remark that if $N$ is small, tychastic and Cox & Rubinstein uncertainty lead to similar evaluation but when $N$ goes to infinity, the tychastic evaluation converges whereas the Cox & Rubinstein model diverges (see figure 17 when $N = 1000$). Indeed in presence of transaction costs, in the tychastic case the portfolio is not often revalued and in the Cox & Rubinstein case it must be revalued at each step as we can see on figure 12 which represents the evolution of the control $u(t, S_1)$ in both situations.

In the lack of fix transaction cost, when $\beta = 0$, for larger values of $N$ the following examples show a strong regularity of the value function with respect to time $t_n : 1, ..., N$, for $N = 200$ fixed. (figure 9).

3.5.1. Proportional and fix transaction costs

In this situation, the method we have presented above cannot be applied directly since in the presence of fix transaction costs the problem falls within the study of impulse dynamical games. Indeed, from a mathematical point of view, the right hand side of the dynamical system which describes the evolution of the financial items is lower semicontinuous. However, as we will see in the last section, this problem can be studied in the frame of impulse systems since whenever $p(t)$ is non null, that is to say that a transaction occurs at time $t$, the trajectory $(x(t), p_1(t), y(t))$ is reset on a new position $(x(t), p_1(t), y(t) + \beta_1)$.

**Algorithm II** (with transaction cost and terminal adjustment)

The guaranteed evaluation function $W$ depends on $S_1$ and $p_1$. 


Viable Capture Basin for Studying Differential and Hybrid Games...

Let us consider the minimal value of $W_\rho(T, S_1, p_1)$ with respect to $p_1 \in [0, 1]$ and the Capital Minimal function

$$S_1 \mapsto W_\rho(T, S_1) = \min_{p_1 \in [0, 1]} W_\rho(T, S_1, p_1) = W_\rho(T, S_1, \bar{p}_1(T))$$

![Graph of $W_\rho(T, S_1, p)$ and Capital Minimal function $W_\rho(T, x)$](image)

Fig. 10. The graph of $W_\rho(T, S_1, p)$ and the Capital Minimal function $W_\rho(T, x)$.

The Argmin value of the function $(T, S_1, p_1) \mapsto W_\rho(T, S_1, p_1)$ provides the optimal rule for the constitution of the replicating portfolio if the value of the share is $S_1$ and the maturity is $T$.

Figure 10 shows the graph of the map $(S_1, p) \mapsto W_\rho(T, S_1, p)$.

Figure 11 represents the projection on the plane $(S_1, W)$ of the graph of the Capital Minimal function $W_\rho$.

![Projection on $(S_1, W)$](image)

Fig. 11. The Capital Minimal function $x \mapsto W_\rho(T, x)$.

Figure 12 shows the optimal buy & sell strategy $u^* = \gamma_\rho(n \rho, S_1, p)$ which minimizes at each step $n$ the right hand side of the equation defining $W_\rho(n + 1, S_1, p)$.

Figure 13 shows variations of function $W_\rho(T, S_1, p)$ with respect to $\alpha_1$ for different values of $N \in \{10, 20, 30, 40, 50, 60, 70, 80, 90\}$.
Fig. 12. Evolution of the optimal buy & sell strategy ($\rho, u$) for the Cox-Rubinstein model.

Fig. 13. Variations of the evaluation function with respect to $\alpha_1$ for stochastic uncertainty $\sigma$, when $N$ varies from 10 to 90.
Figure 14 shows variations of function $W(T, S_1, p)$ with respect to the rate of transaction cost $\alpha_1 \in \{0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10\}$ when $N$ varies from 1 to 125.

Fig. 14. Variations of the evaluation function with transaction cost with respect to $N$ for different values of $\alpha_1 \in \{0, 0.01, ..., 0.1\}$ when $N$ varies from 1 to 125.

**ALGORITHM III** (Uncoupling Algorithm)

From a numerical point of view, the major difference between the Capture Basin Algorithm designed for approximating the Value Function in the lack of transaction cost (presented in (“Pujal & Saint-Pierre (2001)*”)), and the Capture Basin Algorithm for Transaction Cost (presented in “Aubin, Pujal & Saint-Pierre” (2001)) lies in the larger number of variables involved.

The valuation function $W$ is defined on the set $\{0, \rho, ..., N\rho\} \times R^+ \times [0, 1]$. When $n = 0$, the state $(0, x^0, p^0)$ must have reached the target $Epigraph(c)$.

Let us assume that at maturity the replicating portfolio must not contain any quantity of securities. This amounts to impose $p^0 = 0$. If not there will be a final transaction cost to be added to the value of the call since, is the case, the seller of a call for instance will have to sell the risky part, if positive, of the replicating portfolio which will entail transaction cost. Moreover we assume that functions $b$ and $c$ do not depend on $p$.

Thank to this assumption, we can define parallel sequences of functions $W^n_\rho$ and $P^n_\rho$ defined on $R \times R^+$ with values in $R^+$ satisfying $W^n_\rho(S_1) = c(0, S_1)$. Since $p + pu = p^0 = 0$ and $p \in [0, 1]$, at the first step we determine for any $S_1$ the minimal

---

$^3$Computing the Value function $W(t, S, p)$ needs roughly $4N_t, N_S, N_p, \text{octets}$ where $N_t = N, N_s$ and $N_p$ the grids size for representing $S_1$ and $p_1$. For instance, the memory space required for implementing the second algorithm is $500Mo$ if $N_t = N_S = N_p = 500$ and $4Go$ if $N_t = N_S = N_p = 1000$. For implementing the first algorithm the memory space required is $4Mo$ if $N_t = N_S = 1000$. 

---
We can notice the central roles played by the two types of uncertainty: the tychastic uncertainty corresponds to a lower constant value and we set $P^1_\rho(S_1) = -\rho\tilde{u}^0(S_1)$.

In other words, at step one, if the value of the risky asset is $S_1$ then the replicating portfolio contains $P^1_\rho(S_1)$ quantities of this asset and the its value is equal to $W^1_\rho(S_1)$.

At the following steps we define in the same way from $W^1_\rho(S_1)$ and from $P^\ast_\rho(S_1)$:

$$W_{\rho}^{n+1}(S_1) := \max_{\rho(S_1 + ) = [\rho(S_1 + )]} \sup_{u \in [-\frac{1}{\rho}]} \inf_{v \in [v_m, v_M]} W_{\rho}^n(S_1(1 + \rho\gamma_1(S_1, v))) - \rho(u\alpha_1S_1) + \rho(\alpha_1S_1) + \rho(\alpha_1S_1)$$

and

$$P_{\rho}^{n+1}(S_1) := P_{\rho}^n(S_1) + \rho\tilde{u}^n(S_1)$$

where $\tilde{u}^n(S_1)$ is the minimal argument which defines $W_{\rho}^{n+1}(t, S_1)$.

Remark: in general equality between $W_{\rho}^n(S_1)$ and $W_{\rho}(\rho(S_1 + ) = [\rho(S_1 + )]$ does not hold.

3.6. Application: Sensibility with respect to the type of uncertainty

Between tychastic and stochastic process there exists a wide spectrum of different type of uncertainty which can be represented by the choice of $\lambda$ in the definition of the uncertainty range on small time intervals of length $\rho = \frac{1}{\lambda}$: $v \in [v_m, v_M] = [e^{-\sigma \rho \lambda}, e^{-\sigma \rho \lambda} - 1]$. Let us point out that tychastic uncertainty corresponds to $\lambda = 1$ and Cox & Rubinstein uncertainty corresponds to $\lambda = \frac{1}{2}$.

Numerical observations When $N$ increases, the value of $W_{\rho}^n(S_1)$ jumps from a lower constant value $W_m = 5$ to an upper constant value $W_M = 48$ if $\lambda = 1$ but, if $\lambda = \frac{1}{2}$ it remains to a lower value $W_m = 14$ and explodes when $N$ becomes sufficiently high.

Analyzing values given in table 15 looking at figures 16, 17, 18, 19, 20 and 21 we can notice the central roles played by the two types of uncertainty: the tychastic one when $\lambda = 1$ and the Cox & Rubinstein one when $\lambda = \frac{1}{2}$.

- 1) if $\lambda = 1$ the value of the call first decreases then jumps from a minimal value to $W = 46.7$. Then it remains constant.
- 2) if $\lambda > 1$, uncertainty is super-tychastic, the value of the call first decreases and then jumps sooner to a lower value than $W$. Then it continually decreases.
### Table 1

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Fig. 15. Evolution “at the value” (\(S_1 = K\)) of the value of a call when \(N\) varies from 1 to 1000. Comparative results with respect to \(\lambda\).
3) if $\frac{1}{2} < \lambda < 1$, uncertainty is *sub-tychastic and super-CR-stochastic*, the value function is first greater than the tychastic reference and it jumps later to a value greater than $\overline{W}$. Then it continually increases.

4) if $\lambda = \frac{1}{2}$, uncertainty is *CR-stochastic*, the value of the call is constant $\overline{W} = 15.2$ on interval $n \in [1, \overline{W}]$. This value $\overline{W}$ is lower than $\overline{W}$ it suddenly explodes as soon as $n$ becomes greater than $\overline{N}$ ($W = 964602$).

5) if $\lambda < \frac{1}{2}$, uncertainty is *sub-CR-stochastic*, the value of the call is first decreasing over $\overline{W}$ and it jumps to a greater value than the previous one. Then it becomes definitely “incommensurable”...

6) when $\lambda = 0$, that is to say when the uncertainty range is independent from $N$, the value of the call increases but we observe a paradox phenomenon: the growth is no more sudden. A question arises: does it exists a finite limit value when $N \to \infty$?

4. Fix transaction costs and dividends, a hybrid system

Dynamical impulse system describes the evolution of a state variable $x \in X = \mathbb{R}^n$ which, in response to some events, may switch between a continuous evolution and an impulse evolution. Switches are triggered when the state reaches a closed set $C$. During some periods the state is governed by a continuous evolution until it reaches some state $x \in C$ where a reset to a new position $x^\dagger \in \Phi(x)$ may occur. The problem of hedging portfolio clearly belongs to this class of systems. In this section we emphasize this point of view.

Let us briefly recall the general concepts relative to hybrid systems.
Fig. 17. Variation “at the Value” with respect to uncertainty: comparison for $N \in [1, 1001]$: Curves 1: $\lambda = \frac{1}{4}$; 2: $\lambda = \frac{1}{2}$; 3: $\lambda = \frac{3}{4}$; 4: $\lambda = 1$; 5: $\lambda = \frac{5}{4}$; 6: $\lambda = \frac{6}{4}$; 7: $\lambda = \frac{7}{4}$; 8: $\lambda = 2$; 9: $\lambda = 0$.

Fig. 18. Variation “at the Value” for sub and super-tychastic uncertainty: comparison for $N \in [1, 120]$: Curves 1: $\lambda = 0.80$; 2: $\lambda = 0.85$; 3: $\lambda = 0.90$; 4: $\lambda = 0.95$; 5: $\lambda = 1$; 6: $\lambda = 1.05$; 7: $\lambda = 1.10$; 8: $\lambda = 1.15$; 9: $\lambda = 1.20$. 
Fig. 19. Variation “at the Value” for sub and super-tychastic uncertainty: comparison for $N \in [1, 1001]$
Curves 1: $\lambda = 0.80$ ; 2: $\lambda = 0.85$ ; 3: $\lambda = 0.90$ ; 4: $\lambda = 0.95$ ; 5: $\lambda = 1$ ; 6: $\lambda = 1.05$ ; 7: $\lambda = 1.10$ ; 8: $\lambda = 1.15$ ; 9: $\lambda = 1.20$.

Fig. 20. Variation “at the Value” for sub and super-tychastic uncertainty: comparison for $N \in [1, 120]$
Curves 1: $\lambda = 0.30$ ; 2: $\lambda = 0.35$ ; 3: $\lambda = 0.40$ ; 4: $\lambda = 0.45$ ; 5: $\lambda = 0.50$ ; 6: $\lambda = 0.55$ ; 7: $\lambda = 0.60$ ; 8: $\lambda = 0.65$ ; 9: $\lambda = 0.70$. 
4.1. Hybrid Systems: Definition

Let $K$ be a compact set, $T$ and $C \subset K$ closed subset contained in $K$. Assume that $F$ is Marchaud and that $\Phi$ is an upper semi-continuous set valued-map with compact values.

Hybrid systems are formalized as follows.

1. The continuous evolution is given by the differential inclusion

$$x'(t) \in F(x(t)), \text{ for almost all } t \in \mathbb{R}^+ \quad (4.19)$$

We denote by $S^c_F(x_0)$ the set of all absolutely continuous solutions of (4.19) starting from $x_0$ at time $t_0 = 0$.

2. The impulse evolution is given by the recursive inclusion

$$x^{n+1} \in \Phi(x^n) \quad (4.20)$$

We denote by $S^d_F(x^0)$ the set of all discrete solutions $\{x^0, x^1, ..., x^k\}$ of (4.20) starting from $x^0 \in C$.

**Definition 4.1.** A *run* of an impulse system $(F, \Phi)$ is a sequence of elements $x(\cdot) := \{(\tau_i, x_i, x_i(\cdot))\}_{i \in I} \subset (\mathbb{R}^+ \times X \times C(0, \infty; X))^N$, where $\tau_i$ is the $i^{th}$ cadence, $x_i$ is the $i^{th}$ reinitialization, with $x_0 = x^0$, and $x_i(\cdot) \in S^c_F(x_i)$ is the $i^{th}$ motive which is an almost continuous solution to (4.19) starting from $x_i$ at time $0$ until time $\tau_i$.

We denote by $S_{(F, \Phi)}(x_0)$ the set of runs starting from $x_0$. The set of indexes $I = \{0, 1, ..., n\} \subset \mathbb{N}$ can be finite ($n < +\infty$) or infinite ($n = +\infty$), satisfying

$$\forall i < n, \exists x_i(\cdot) \in S^c_F(x_i) \text{ such that } x_i(\tau_i) \in C, x_{i+1} \in \Phi(x_i(\tau_i))$$
If $\tau_i = 0$, $x_{i+1} \in \Phi(x_i)$ and then $x_i(\cdot)$ is defined on an interval of length 0.

We set $T = \sum_{i=0}^{n} \tau_i$.

Let $K$ be a closed set. An impulse constrained system is characterized by the triple $(F, \Phi, K)$ and we denote by $S_{(F,\Phi,K)}(x_0)$ the set of runs $\overline{x}(\cdot)$ starting from $x_0$ and viable in $K$.

**Definition 4.2.** The Hybrid Viability Kernel (or Hybrid Kernel) of $K$ for the impulse system $(F, \Phi, K)$ is the largest closed subset of initial states belonging to $K$ from which starts at least one hybrid viable solution. We denote this set $Hyb_{(F,\Phi)}(K)$.

The notion of hybrid kernel has been introduced in ("Aubin (1999)"). It can be characterized in terms of capture basin. As for Capture Basin, one can prove that the hybrid kernel is closed and that hybrid Capture Basins can be approximated by a sequence of discrete viability kernels associated with suitable discrete systems (see "Saint-Pierre (2001)").

We can extend the Guaranteed Capture Basin algorithm to approximate Guaranteed Hybrid Capture Basins and we apply it to the problem of hedging portfolio in presence of dividends and transaction costs.

### 4.2. Application 1: Hedging Portfolio with Dividends

Let $d$ the dividend payed at some date $T_0 < T$.

Let us consider the reset set

$$C = \{(t, S_1, p_1, W)|t = T_0\}$$

and the reset map

$$\Phi(t, S_1, p_1, W) = (t - \delta, S_1 - d, p_1, W + p_1d)$$

where $\delta > 0$ is a virtual laps of time which indicates that a reset occurred.

We define the hybrid system

$$(t', S_1', p_1', W') \in \{-1\} \times F(S_1, p_1, W), \text{ a.e. } t$$

$$(t^{n+1}, S_1^{n+1}, p_1^{n+1}, W^{n+1}) = \Phi(t^n, S^n, W^n), \ t = T_0$$

Since the Impulse Kernel $Hyb_{(1)}(F,\Phi)(K)$ is empty, the capture domain of the epigraph of $u$ coincides with $Hyb_{(1)}(F_C,\Phi)(K)$ where $F_C = F \circ C^c$ and $F_C = 0$ on $C$.

$$Capt_{F_C,\Phi}(Epi(u)) = \overline{Viab}_{G^c}(K)Hyb_{(1)}(F_C,\Phi)(K)$$

We consider the discrete model define above and the Algorithm II.

**Exemple:**

Let us consider the basic European call for Cox & Rubinstein type of uncertainty and assume that the payment of dividend is done at time $T_0 = 0.1$, the strike is $K = 100$ and the maturity time is $T = 1$. We get the following numerical values which correspond to values given by standard methods.
4.3. Application 2: Hedging Portfolio with Transaction Costs

But we can also interpret the evaluation problem of a call in term of impulse control system under uncertainty. Since transaction cost may now appear at any time, the reset set is the whole space \( C = \{ (t, S_1, p_1, W) \in [0, T] \times R^+ \times [0, 1] \times R^+ \} \) and the reset set-valued map becomes

\[
(t, S_1, p_1, W) \rightarrow \Phi(t, S_1, p_1, W) := \{(t, S_1, q, W + |p_1 - q|\alpha_1 S_1), \quad q \in [0, 1]\}
\]

We define formally the (continuous) hybrid system

\[
\begin{align*}
(t^{n+1}, S_1^{n+1}, p_1^{n+1}, W^{n+1}) & \in \{ -1 \} \times F(S_1, p_1, W), \quad \text{a.e. } t \in [0, T] \\
(t^n, S_1^n, p_1^n, W^n) & \in \Phi(t^n, S_1^n, p_1^n, W^n)
\end{align*}
\]

Then considering the discrete dynamical system associated with (4.21) we have

\[
\begin{align*}
t^{k+1} &= t^k - \mu \rho \\
S_1^{k+1} &= S_1^k (1 + \gamma_\rho (S_1^k, \nu)), \quad i = 0, 1 \\
p_1^{k+1} &= p_1^k + (1 - \mu) \rho \nu \\
W^{k+1} &= W^k (1 + \gamma_\rho (S_1^k)) + p_1^k S_1^k (\gamma_\rho (S_1^k, \nu) - \gamma_\rho (S_0^k) - (1 - \mu) \rho |u| \alpha_1 S_1^k - (1 - \mu) \beta_i)
\end{align*}
\]

where \( u \in [-\frac{\rho}{\gamma_\rho}, 1 - \frac{\rho}{\gamma_\rho}] \) and \( \mu \) belongs to the discrete set \( \{0, 1\} \).

If \( \mu^k = 0 \) a transaction is effective at time \( n = t^k \). If \( \mu^k = 1 \), there is no transaction at time \( n = t^k \).

References

BERNHARD P. (2002) Robust control approach to option pricing, including coût de transaction, Annals of Dynamic Games

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