

THE COUPLING OF CLIMATE AND  
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Essays on Integrated  
Assessment

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## Chapter 1

# A VIABILITY APPROACH TO GLOBAL CLIMATE CHANGE ISSUES

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**Abstract** The main purpose of viability theory is to explain the evolution of the state of a control system governed by nondeterministic dynamics and subjected to viability constraints, reveal the concealed feedbacks which allow the system to be regulated and provide selection mechanisms for implementing them. It assumes implicitly an “opportunistic” and “conservative” behavior of the system: a behavior which enables the system to keep viable solutions as long as its potential for exploration (or its lack of determinism) — described by the availability of several evolutions — makes its regulation possible. It also happens that these results can be used to study infinite horizon optimal control problems, with intergenerational constraints, with nonstandard inter-temporal optimality criteria bearing not only on the evolutions of the states and the controls, but also on the velocities of the controls, allowing us in particular to minimize maximal inertia. We illustrate these points with simplified Greenhouse Gas models, where we minimize the worst transition cost of changing the short-term pollution rate (or economic growth) in order to maintain the concentration of greenhouse gases bounded.

### 1. A short Introduction to Viability Theory

This paper complements the infinite horizon optimization viewpoint exposed in Haurie, 2003, for instance. We shall add to the infinite horizon optimization paradigm intergenerational constraints which may be relevant to the questions of global climate change. We shall also study inertia functions which we presented in the preceding section, which are other infinite horizon optimal problems where we minimize the worst cost of the derivative of the control. The concepts and tools used for this purpose come from viability theory, and in particular, the concept of viability kernel. These concepts may be useful not

only as mathematical tools for studying and solving numerically optimal control problems, but also interesting by themselves for providing mathematical metaphors and tools for some climatic issues as well. Indeed, contrary to optimal control theory, viability theory does not require any single decision-maker (or actor, or player) to guide" the system by optimizing an intertemporal optimality criterion<sup>1</sup>. Furthermore, the choice (even conditional) of the controls is not made once and for all at some initial time, but they can be changed at each instant so as to take into account possible modifications of the environment of the system, allowing therefore for adaptation to viability constraints. Finally, by not appealing to intertemporal criteria, viability theory does not require any knowledge of the future<sup>2</sup> (even of a stochastic nature.) This is of particular importance when experimentation<sup>3</sup> is not possible or when the phenomenon under study is not periodic. For example, in biological evolution as well as in economics and in the other systems involving life, the systems are irreversible, their dynamics may disappear and cannot be recreated, forbidding any insight into the future. Hence, forecasting or prediction of the future are not the issues which we shall address in viability theory.

However, the conclusions of the theorems allow us to reduce the choice of possible evolutions, or to single out impossible future events, or to provide explanation for some behaviors which do not fit any reasonable optimality criterion. Therefore, instead of using intertemporal optimization<sup>4</sup> that involves the future, viability theory provides selection procedures of viable evolutions obeying, at each instant, state constraints which depend upon the present or the past. (This does not exclude anticipations, which are extrapolations of past evolutions, constraining in the last analysis the evolution of the system to be a function of its history.)

Viability theory designs and develops mathematical and algorithmic methods for studying the evolution of systems and networks of systems (or organizations, organisms), with the following characteristics:

- under continuous time, discrete time, or an "hybrid" of the two when impulses are involved,
- constrained to adapt to a (possibly co-evolving) environment,

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<sup>1</sup>the choice of which is open to question even in static models, even when multicriteria or several decision makers are involved in the model.

<sup>2</sup>Most systems we investigate do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

<sup>3</sup>Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to know" the future evolution of the system.

<sup>4</sup>which can be traced back to Sumerian mythology which is at the origin of Genesis: one Decision-Maker, deciding what is good and bad and choosing the best (fortunately, on an intertemporal basis, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week...

- evolving under contingent, stochastic or tychastic uncertainty. State-dependent uncertainty can also be translated mathematically by parameters on which actors, agents, decision makers, etc. have no controls. These parameters are often perturbations, disturbances (as in “robust control” or “differential games against nature”) or more generally, *tyches* (meaning “chance” in classical Greek, from the Goddess Tyche) ranging over a state-dependent tychastic map. They could be called “random variables” if this vocabulary were not already confiscated by probabilists. This is why we borrow the term of *tychastic evolution* to Charles Peirce who introduced it in a paper published in 1893 under the title *evolutionary love*. One can prove that stochastic viability is a (very) particular case of tychastic viability. The size of the tychastic map captures mathematically the concept of “versatility (tychastic volatility)” — instead of “stochastic volatility”: The larger the graph of the tychastic map, the more “versatile” the system.
- using for this purpose *regulons* (regulation controls), and in the case of networks, connectionist matrices or tensors,
- regulated by feedback laws (static or dynamic) that are then “computed” according to given principles, such as the inertia principle,
- co-evolving with their environment (*mutational viability*),
- the nonviable dynamics being corrected by introducing adequate controls (*viability multipliers*) when necessary.

We introduce the viability kernel with target under a nonlinear controlled system (either continuous or hybrid): See Figure 1.1. This is the subset of initial states from which starts at least one evolution that either (i) remains in the constrained set (i.e., is viable) forever; or (ii) reaches (i.e., captures) the target in finite time<sup>5</sup> before possibly violating the constraints. When the target is empty, only the first condition matters, and one says that it is simply the viability kernel of the environment. The set of initial states satisfying only the second condition is called the capture basin of the target viable in the constrained subset.

When these evolutions depend upon a parameter, such parameter can be regarded as (i) a control when actors (agents, decision makers, etc.) can act (pilot, decide, choose, etc.) on them; (ii) as regulatory parameters, in short a *regulons*, when no clearly identified agent can act on them, these parameters are regarded as genotypes in biology, fiduciary goods in economics, cultural codes in sociology. They range over a state-dependent cybernetic map, providing the system opportunities to adapt to viability constraints (often, as slowly as possible) and/or to regulate intertemporal optimal evolutions.

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<sup>5</sup>and not only asymptotically, as it is usually studied with concepts of attractors since the pioneering works of Alexander Lyapunov and Henri Poincaré going back to 1892.

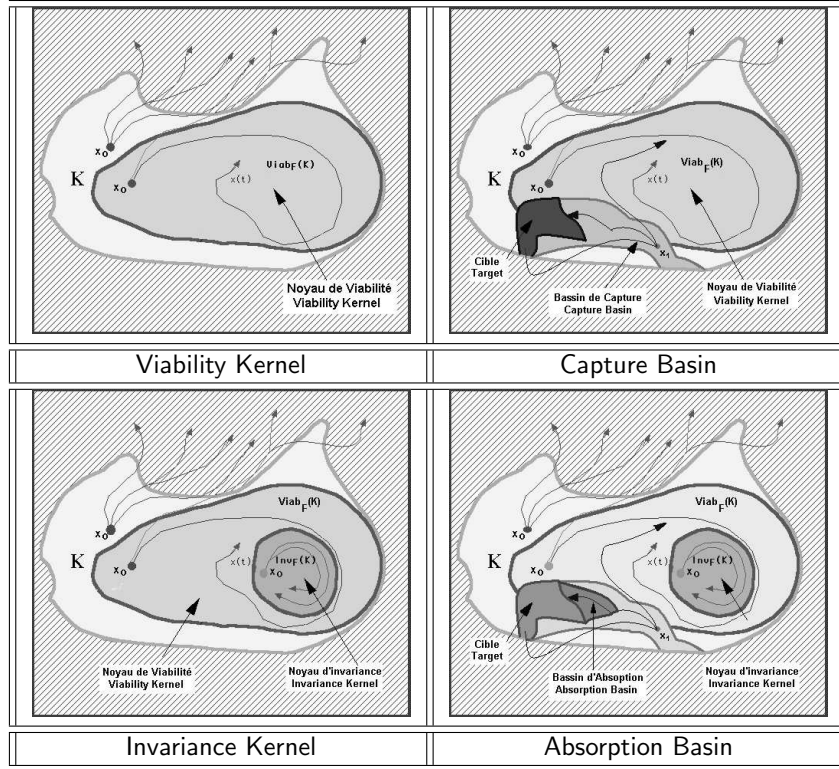


Figure 1.1. Kernels and Basins.

We also introduce the “dual” concept of invariance kernel with target, which is the subset of initial states from which all evolutions either (i) remain in the environment (i.e., are viable) forever or (ii) reach the target in finite time before possibly violating the constraints. The set of initial states satisfying only the second condition is called the absorption basin of the target invariant in the environment. This concept plays a role whenever the evolutions are governed by evolutions depending upon tyches (perturbations, disturbances) on which actors, agents, decision makers, etc.

Tychastic control systems (or dynamical games) involve both regulons and tyches in the dynamics, tyches describing uncertainties played by an indifferent, maybe hostile Nature, regulons being available and chosen by the system in order to adapt its evolutions whatever the tyches. Let us introduce the concepts of tychastic (or guaranteed) viability kernel, which is the subset of initial states from which there exist a regulon such that, for all tyches, the associated evolutions either (i) remain in the environment (i.e., are viable) forever or (ii) reach the target in finite time before possibly violating the constraints. The set of initial states satisfying only the second condition is called the absorption basin of the target invariant in the environment.

It is by now a consensus that many variables describing systems, organizations, networks arising in biology and human and social sciences do not evolve in a deterministic way, and may be, not even in a stochastic way as it is usually understood, but with a Darwinian flavor, where intertemporal optimality selection mechanisms are replaced by several forms of "viability", a word encompassing polysemous concepts as stability, confinement, homeostasis, tolerable windows<sup>6</sup> etc., expressing the idea that some variables must obey some constraints: We quote Petschel-Held et al., 1999:

*The tolerable windows approach (TWA) allows the climate policy formulation process to be safeguarded in the following way. First, guardrails are defined in order to exclude intolerable climate change impacts, on the one hand, and unacceptable socioeconomic consequences of climate change mitigation measures, on the other. Second, a scientific analysis is conducted to investigate the features of those emission paths that are compatible with the guardrail constraints. The fundamental methodology of the TWA is best described in terms of the theory of differential inclusions...*

Intertemporal optimization is replaced by myopic selection mechanisms that involve present knowledge, sometimes the knowledge of the history (or the path) of the evolution, instead of anticipations or knowledge of the future (whenever the evolution of these systems cannot be reproduced experimentally). Uncertainty does not necessarily obey statistical laws, but only unforecastable rare events (tyches, or perturbations, disturbances) that obey no statistical law, that must be avoided at all costs (precautionary principle or tychastic (robust) control). These systems can be regulated by using regulation (or cybernetical) controls that have to be chosen as feedbacks for guaranteeing the viability of a system and/or the capturability of targets and objectives, possibly against tyches (perturbations played by Nature).

**Outline:** In the next section 2 we motivate the Viability approach through two simple examples involving ecological constraints, the evolution of greenhouse gas concentration, the evolution of macro-economic interaction and the inertia function which provides the minimal worst effort for regulating pollution. We devote the section 3 to the description of the main concepts and basic results of viability theory: evolutions, viability kernels and capture basins under evolutionary systems. In section 4 we consider specific evolutions by defining inertia functions. We emphasize the relation between these functions and the Viability Theory proving that their epigraphs are viability kernels associated with extended dynamical systems. It follows that the Viability Kernel Algorithm can be implemented for computing approximations of these inertia functions. In section 5 we complements the infinite horizon optimization viewpoint exposed in Haurie, 2003 by characterizing the epigraph of the value function of

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<sup>6</sup>see the contribution by Ferenc, Toth and Petschel-Held et al., 1999, Bruckner et al., 1999, Toth, 2002, etc.

infinite horizon optimal control problems as a viability kernel (epigraphical approach of the Hamilton-Jacobi-Bellman approach). We add to the model some intergenerational constraints which may be relevant to the questions of global climate change. In section 5, we consider another infinite horizon optimal problem where we maximize the worst cost of the derivative of the control. This provides a simple example of minimizing the worst transition cost of changing the pollution rate in order to keep the concentration of greenhouse gas constant. In the last section 6 we characterize, in the viability/capturability frame, the adaptation law for the construction of static dynamic feedbacks.

## 2. Motivation: Inertia Functions of Simple Examples

Many difficulties of collective decision-making models about climatic risks are due to the interactions between physical and economical requirements. The precautionary principle<sup>7</sup> and economic efficiency are often contradictory<sup>8</sup>.

For stylizing the problems and providing a two-dimensional illustration, we begin by isolating two variables:

- the concentration  $x(t) \in [0, b]$  of greenhouse gases — say,  $CO_2$  — regarded as a state variable, bounded by a given constant  $b$
- the short-term pollution rate (generated when using a given technology and a level of production)  $y(t) \in \mathbb{R}_+$ , regarded as controls or regulons (regulatory controls).

The *ecological constraints* being represented by the interval  $[0, b]$ , the *economic constraints* amounts to bound or minimize a transition cost measured in this example by the absolute value of the velocity of the pollution rate. How this cost is a new constraint for a macro-economic model is another question which we examine next.

We assume here that the evolution of the concentration of greenhouse gases is governed by the differential equation

$$x'(t) = y(t) - ax(t).$$

That means that the variation of the concentration of the greenhouse gases depends upon a natural slow absorption phenomenon by the oceans ( $-ax(t)$  with a “small”) and is proportional to the short-term pollution.

This starting example is chosen for providing analytical solutions. The Saint-Pierre viability kernel algorithm which we shall use later allows nonlinear differential equations that climatologists can propose, such as the ones used by

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<sup>7</sup>stating that one should limit, bound or even forbid potential dangerous actions, without waiting for a scientific proof of their hazardous consequences, whatever the economic cost.

<sup>8</sup>see Aubin, 1996b for certain comments on this topic; we also refer to Doyen et al., 1996; Petschel-Held et al., 1999; Bruckner et al., 1999; Doyen & Gabay, 1999; Doyen & Gabay, 1997; Doyen & Gabay, 1996; Gabay 1994.

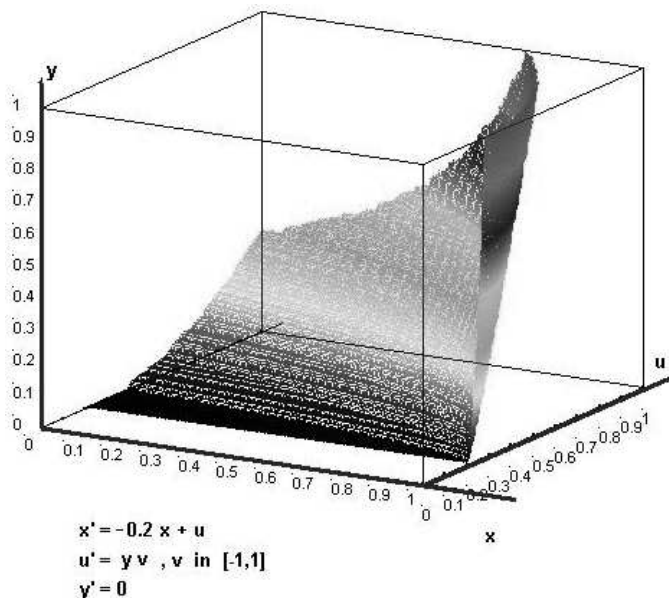


Figure 1.2. Graph of the Inertia Function  $\alpha(x, y) := \inf_{(x(\cdot), y(\cdot)) \in \mathcal{P}(x, y)} \sup_{t \geq 0} |y'(t)|$  where  $x(\cdot)$  is governed by  $x'(t) = y(t) - 0.2x(t)$  defined on  $[0, 1] \times [0, 1]$  (instead of  $[0, 1] \times \mathbb{R}_+$ ). The variable  $y$  is the pollution rate, the state  $x$  is the concentration of greenhouse gases and the inertia function as the minimaximal intertemporal transition cost of changing of pollution rate.

the German Advisory Council on Global Change (WBGU) in its special report for the First Conference of the Parties to the FCCC in Berlin (WBGU, 1997). A forthcoming study by T. Bernado, P. Saint-Pierre and J. Scheffran will adapt some of the methods presented here in the framework of such models.

We denote by  $\mathcal{P}(x, y)$  the set of evolutions  $(x(\cdot), y(\cdot))$  starting at  $x(0) = x$  and  $y(0) = y$ , viable in  $[0, b] \times \mathbb{R}_+$  where  $x(\cdot)$  is governed by  $x'(t) = y(t) - ax(t)$ .

The inertia function that we shall study in this paper is defined by

$$\alpha(x, y) := \inf_{(x(\cdot), y(\cdot)) \in \mathcal{P}(x, y)} \sup_{t \geq 0} |y'(t)|.$$

Starting with initial greenhouse gas concentration  $x$  and emission  $y$ , the inertia function provides the minimal worst effort measured by the velocity of the regulon with which the pollution emission should change in order for the greenhouse gas concentration to satisfy the viability constraints  $x(t) \in [0, b]$ . This in an infinite horizon optimal control problem which is non-standard in at least two respects: (i) The intertemporal optimality criterium bears on the velocities of the controls (and not only ont the evolutions of the states and the controls, as it is usually the case); (ii) The criterion is not as usual an integral criterion, but a supremum over time of a Lagrangian function (here, the norm).

One can compute this inertia function with the viability kernel algorithm as we shall explain later (see Figure 1.2) and also compute it explicitly since  $\alpha(x, y)$  is the solution to the equation

$$a(x - ay) = \alpha(x, y) \left( 1 - e^{\frac{a}{\alpha(x, y)}(y - ab)} \right).$$

The warning function  $\Xi_c(y) := \{x \in [0, b] \mid \alpha(x, y) = c\}$  is a single-valued function  $\mathbb{R}_+ \mapsto \mathbb{R}$  given by

$$\Xi_c(y) = \frac{c}{a^2} \left( 1 - e^{\frac{a}{c}(y - ab)} \right) + \frac{y}{a}.$$

It satisfies  $\Xi_c(ab) = b$  and  $\xi_c(0) = \frac{c}{a^2} \left( 1 - e^{\frac{-a^2 b}{c}} \right)$ , which provides the smallest concentration of greenhouse gas that we can obtain by choosing the most drastic reduction strategy using  $y'(t) = -c$ .

We observe that  $\alpha(x, y) = 0$  if and only if  $(x, y) \in [0, b] \times [0, ab]$ . In this case, passive evolutions  $x_y(\cdot)$  are viable in  $[0, b]$  and converge to the equilibrium  $\frac{y}{a}$ , which is stable whenever  $y \in [0, ab]$ . The level sets are defined by the formula

$$\{(x, y) \in [0, b] \times \mathbb{R}_+ \mid \alpha(x, y) \leq c\} = \{(x, y) \in [0, b] \times \mathbb{R}_+ \mid x \leq \Xi_c(y)\}.$$

They provide the subsets of state-control pairs for which it is possible to satisfy the viability constraints with control velocities bounded in norm by a given constant  $c \geq 0$ . The trajectories of evolutions  $(x(\cdot), y(\cdot))$  satisfying  $\alpha(x(t), y(t)) = c$  (called *inert evolutions*) satisfy the equation  $x(t) = \Xi_c(y - ct)$ . The *heavy evolution* consists in keeping the same pollution as long as the mass of greenhouse gas is smaller than  $\Xi_c(y)$ . At this level, the technology has to be drastically changed with the velocity equal to  $-c$ , while the concentration of greenhouse gas increases until it reaches the level  $b$ , which is an equilibrium where the heavy evolution stops.

As a second example, we can add a macro-economic interaction, stating that the emissions of pollutants depend upon the economic<sup>9</sup> activity  $z(t)$ . Hence, at a very elementary level of illustration of the phenomenon, we assume that we have access to the velocity of the pollution rate through a bound of the form  $|y'(t)| \leq z(t)$  set by economic activity  $z(t)$ . This ignorance is taken into account by the “meta-inertia function” that we now define.

We denote by  $\mathcal{P}(x, y, z)$  the subset of “meta-evolutions”  $(x(\cdot), y(\cdot), z(\cdot))$  starting at  $x(0) = x$ ,  $y(0) = y$  and  $z(0) = z$ , viable in  $[0, b] \times \mathbb{R}_+ \times \mathbb{R}$  where  $(x(\cdot), y(\cdot))$  is governed by

$$\begin{cases} (i) & x'(t) = y(t) - ax(t) \\ (ii) & |y'(t)| \leq z(t). \end{cases}$$

<sup>9</sup>At least one of the authors is not convinced by the validity of the standard macro-economic models, too simplistic to take into account socio-psycho-economic... behaviors. This issue is well known, and has been tackled in many articles such as the Nordhaus Integrated Assessment Models.

The “meta-inertia function”  $(x, y, z) \mapsto \beta(x, y, z)$  associated with this meta-system is defined by

$$\beta(x, y, z) := \inf_{(x(\cdot), y(\cdot), z(\cdot)) \in \mathcal{P}(x, y, z)} \sup_{t \geq 0} |z'(t)|.$$

Since the graph of this function is 4-dimensional, we shall represent it by its 3-dimensional level-sets

$$\{(x, y, z) \in [0, b] \times \mathbb{R}_+ \times \mathbb{R} \text{ such that } \beta(x, y, z) \leq c\}.$$

We observe that the inertia function  $\alpha$  is related to the meta-inertia function  $\beta$  by the relations

$$\text{Graph}(\alpha) = \{(x, y, z) \in [0, b] \times \mathbb{R}_+ \times \mathbb{R} \text{ such that } \beta(x, y, z) \leq 0\}.$$

Indeed, we remark that the inertia  $z := \alpha(x, y)$  of an evolutions  $(x(\cdot), y(\cdot)) \in \mathcal{P}(x, y)$  is finite if and only if, setting  $z(t) \equiv z$ , the meta-evolution  $(x(\cdot), y(\cdot), z(\cdot)) \in \mathcal{P}(x, y, z)$  satisfies

$$\beta(x, y, z) := \inf_{(x(\cdot), y(\cdot), z(\cdot)) \in \mathcal{P}(x, y, z)} \sup_{t \geq 0} |z'(t)| = 0. \blacksquare$$

### 3. The Mathematical Framework

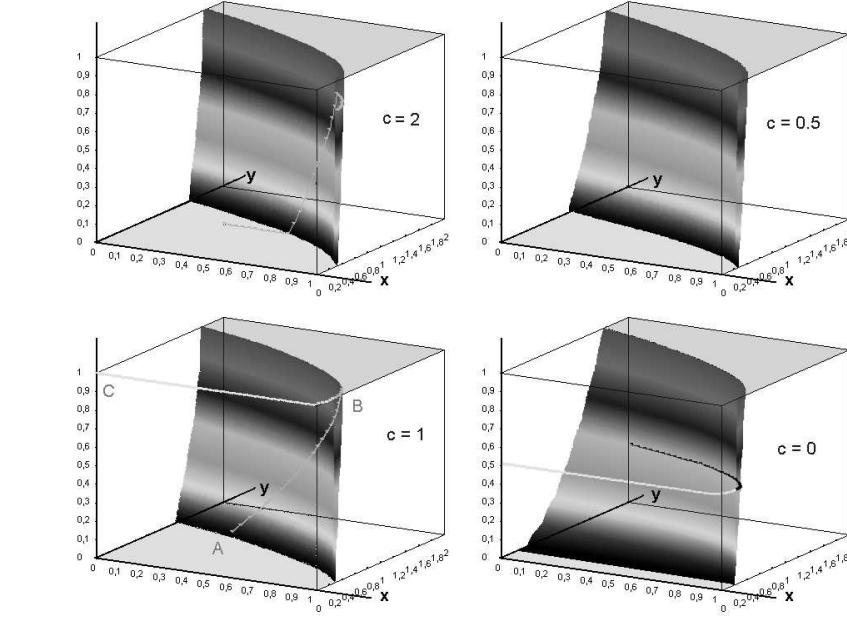
For more details on viability theory, we refer to Aubin, 1991 and to the forthcoming book by Aubin et al., to appear. We provide here basic definitions and some results which may be relevant to climate studies.

#### 3.1 Viability and Capturability

Let  $X$  denote the **state space** of the system. Evolutions describe the behavior of the state of the system as a function of time  $t \in \mathbb{R}_+ := [0, \dots, +\infty[$  ranging over the set of nonnegative real numbers or scalars  $t \in \mathbb{R}_+$ . We shall assume all along that (i) the state space is a finite dimensional vector space  $X := \mathbb{R}^n$ ; (ii) the evolutions are *continuous* functions  $x(\cdot) : t \in \mathbb{R}_+ \mapsto x(t) \in X$  describing the evolution of the state  $x(t)$ . We denote the space of continuous evolutions  $x(\cdot)$  by  $\mathcal{C}(0, \infty; X)$  or, in short,  $\mathcal{C}(X)$ .

Some evolutions, mainly motivated by physics, are classical: *equilibria and periodic evolutions*. But these properties are not necessarily adequate for problems arising in economics, biology, cognitive sciences and other domains involving living beings. Hence we add the concept of evolutions *viable in a constrained set*  $K \subset X$  (the environment) or *capturing a target*  $C \subset K$  in finite time to the list of properties satisfied by evolutions. Therefore, we consider mainly evolutions  $x(\cdot)$  *viable in a subset*  $K \subset X$  representing a constrained set (an environment) in which the trajectory of the evolution must remain forever:

$$\forall t \geq 0, x(t) \in K. \tag{1.1}$$



*Figure 1.3. Level-sets of the Meta-Inertia Function  $\beta(x, y, z) := \inf_{(x(\cdot), y(\cdot), z(\cdot)) \in \mathcal{P}(x, y, z)} \sup_{t \geq 0} |z'(t)|$  when the evolution  $x(\cdot)$  is governed by the system  $x'(t) = y(t) - 0.2x(t)$  and  $|y'(t)| \leq z(t)$  defined on  $[0, 1] \times [0, 25] \times [-1, +1]$  for several values of  $c = 0, 0.5, 1$  &  $2$ . We assume only that the derivative  $y'(\cdot)$  of the pollution rate  $y(\cdot)$  is bounded by a measure of the economic activity. The meta-inertia function provides the minimaximal intertemporal economic transition cost of changing of pollution rate. For  $c = 0$ , we find the graph of the inertia function  $\alpha$ , equal to the level set  $\{(x, y, z) \in [0, b] \times \mathbb{R}_+ \times \mathbb{R} \text{ such that } \beta(x, y, z) \leq 0\}$ . The trajectory of the evolution represented for  $c = 2$  is an heavy evolution (for  $z := \alpha(x, y)$ ) and the evolution represented for  $c = 0$  is an inert one. For  $c = 1$ , the trajectory of the evolution starting from  $B$  is an heavy evolution of the level set  $\{(x, y, z) \in [0, b] \times \mathbb{R}_+ \times \mathbb{R} \text{ such that } \beta(x, y, z) \leq 1\}$  arriving at equilibrium  $A$ .*

Alternatively, a “target”  $C \subset K$  being given, we distinguish evolutions  $x(\cdot)$  capturing the target  $C$  in the sense that they are viable in  $K$  until they reach the target  $C$  in finite time:

$$\exists T \geq 0 \text{ such that } \begin{cases} x(T) \in C \\ \forall t \in [0, T], x(t) \in K. \end{cases} \quad (1.2)$$

We devote our paper to the study of the set of evolutions viable in  $K$  outside  $C$ , i.e. that are viable in  $K$  forever or until they reach the target  $C$  in finite time.

### 3.2 The Evolutionary System

Next, we provide the mathematical description of one of the “engines” governing the evolution of the state. We assume that there exists a control parameter, or, better, a regulatory parameter, called a **regulon**, that influences the evolution of the state of the system. This dynamical system takes the form of a control system with (multivalued) feedbacks :

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \text{ (action)} \\ ii) & u(t) \in U(x(t)) \text{ (contingent retroaction)} \end{cases} \quad (1.3)$$

taking into account the *a priori* availability of several regulons  $u(t) \in U(x(t))$  chosen in a subset  $U(x(t)) \subset Y$  of another finite dimensional vector-space  $Y$  subjected to **state-dependent constraints**. Once the initial state is fixed, the first equation describes how the regulon acts on the velocities of the system whereas the second inclusion shows how the state (or an observation on the state) can **retroact through** (several) regulons in a multivalued way.

We observe that there are many evolutions starting from a given initial state  $x_0$ , one for each time-dependent regulon  $t \mapsto u(t)$ . The set-valued map  $U : X \rightsquigarrow Y$  also describes the **state-dependent constraints** on the regulons. In this case, the system (1.3) can no longer be regarded as a parameterized family of differential equations, as in the case when  $U(x) \equiv U$  does not depend upon the state, but as a **differential inclusion** (see Aubin & Cellina, 1984 for example). Fortunately, differential inclusions enjoy most of the properties of differential equations. A solution to system (1.3) is an evolution  $t \rightarrow x(t)$  satisfying this system for some (measurable) open-loop control  $t \rightarrow u(t)$  (almost everywhere).

We associate with the control system the **evolutionary system**  $x \rightsquigarrow \mathcal{S}(x)$  associating with any initial state  $x \in K$  the subset  $\mathcal{S}(x) \subset \mathcal{C}(0, \infty; X)$  of solutions starting at  $x$ . Most of the results on viability kernels and capture basins depend upon few properties of this evolutionary system, that are shared by other “engine of evolutions”, such as diffusion-reaction systems, path (or history) dependent systems, mutational equations governing the evolution of compact sets.

### 3.3 Viability Kernels and Capture basins

The problems we shall study are all related to the viability of a constrained subset  $K$  and/or the capturability of a target  $C \subset K$  under the dynamical system modelling the dynamic behavior of the system. So let us introduce: (i) The subset  $\text{Viab}(K)$  of initial states  $x_0 \in K$  such that one solution  $x(\cdot)$  to system (1.3)ii) starting at  $x_0$  is viable in  $K$  for all  $t \geq 0$  is called the **viability kernel** of  $K$  under the control system. A subset  $K$  is a **repeller** if its viability kernel is empty; (ii) the subset  $\text{Capt}(K, C)$  of initial states  $x_0 \in K$  such that the target  $C \subset K$  is reached in finite time before possibly leaving  $K$  by one solution  $x(\cdot)$  to system (1.3)ii) starting at  $x_0$  is called the **viable-capture basin** of  $C$  in  $K$ . A subset  $C \subset K$  such that  $\text{Capt}(K, C) = C$  is said to be **isolated** in  $K$ .

We say that

- a subset  $K$  is *viable* under  $\mathcal{S}$  if  $K = \text{Viab}(K)$ ,
- the subset  $K$  is a *repeller* if  $\text{Viab}(K) = \emptyset$ .

In other words, the *viability* of a subset  $K$  under a control system is a consistency property of the dynamics of the system confronted to the constraints it must obey during some length of time. To say that a singleton  $\{c\}$  is viable amounts to saying that the state  $c$  is an *equilibrium* (equi-libra, equal balance) — also called a fixed point. The trajectory of a periodic solution is also viable.

Contrary to the century-old tradition going back to Lyapunov, we require the system to capture the target  $C$  in *finite time*, and not in an asymptotic way, as in mathematical models of physical systems. However, there are close mathematical links between the various concepts of *stability* and *viability*. For instance, Lyapunov functions can be constructed using tools of viability theory. Or one can prove that the attractor is contained in the viability kernel of an absorbing set under the backward (negative) system. This needs much more space to be described: we refer to Chapter 8 of Aubin, 1991 and Chapter 8 of Aubin, 1997 for more details on this topic. One can also prove that *the viability kernel  $\text{Viab}(K)$  of the subset  $K$  is the largest subset of  $K$  viable under the control system*. Hence, all interesting features such as equilibria, trajectories of periodic solutions, limit sets and attractors, if any, are all contained in the viability kernel.

One can prove that the viability kernel is the **unique** subset  $D \subset K$  viable and isolated in  $K$  such that  $K \setminus D$  is a repeller. If  $K \setminus C$  is a repeller, the capture basin  $\text{Capt}(K, C)$  of  $C \subset K$  is the **unique** subset  $D$  between  $C$  and  $D$  such that  $D$  is isolated in  $K$  and  $D \setminus C$  is locally viable. The viability kernels of a subset and the capture basins of a target can thus be characterized in diverse ways through tangential conditions thanks to the viability theorems. They play a crucial role in viability theory, since many interesting concepts are often viability kernels or capture basins. Furthermore, **algorithms** designed in Saint-Pierre, 1994 allow us to compute viability kernels and capture basins (see also Cardaliaguet et al., (1999) and Quincampoix & Saint-Pierre, 1998). In general, there are no explicit formulas providing the viability kernel and capture basins.

#### 4. The Inertia Functions, Metasystems and Viability Niches

In this section, we consider parameterized system

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in U(x(t)) \end{cases} \quad (1.4)$$

where the set-valued map  $U$  involves implicitly the viability constraints

$$\forall t \geq 0, x(t) \in K := \text{Dom}(U).$$

**Remark:** — Conversely, the viability constraint described by a constrained subset  $K$  under parameterized system (1.4) can be taken into account by introducing the restriction  $U|_K$  of the set-valued map  $U$  to  $K$  defined by

$$U|_K(x) := \begin{cases} U(x) & \text{if } x \in K \\ \emptyset & \text{if } x \notin K. \end{cases}$$

This amounts to studying the system

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in U|_K(x(t)) \end{cases}$$

that is the above parameterized system (1.4) when  $U$  is replaced by  $U|_K$ . ■

We shall devote this section to specific evolutions classified in increasing inertia order.

The most “inert” evolutions are **equilibria**  $(x^*, u^*)$  of the control system, solutions to

$$\begin{cases} (i) & 0 = f(x^*, u^*) \\ (ii) & u^* \in U(x^*) \end{cases}$$

since both the state and the controls do not evolve.

We next distinguish passive evolutions, governed by constant controls:

**DEFINITION 4.1** *Evolutions governed by systems*

$$\forall t \in [t_0, t_1], \quad x'(t) = f(x(t), u)$$

*subjected to viability constraints*

$$\forall t \in [t_0, t_1], \quad x(t) \in U^{-1}(u)$$

*with constant regulon are called **passive evolutions** on the time interval  $[t_0, t_1]$ .*

This is a situation familiar in physics, when the parameters are physical coefficients, that do not change along the evolution. Important properties (set of equilibria, stability or unstability of equilibria) are then studied in terms of such parameters, as in bifurcation theory, catastrophe theory, chaotic behavior, etc. Naturally, passive evolutions may not exist, or exist only on a finite time interval, or the class of passive evolutions is too small to contain solutions to given problems (viability, capturability, optimal controls, etc.).

We then consider evolutions regulated by affine open-loop controls of the form  $u(t) := u + u_1 t$ , nicknamed **ramp controls** in control theory, regulating what we shall call “inert controls”.

**DEFINITION 4.2** *An evolution  $(x(\cdot), u(\cdot))$  is said to be **inert** on a time interval  $[t_0, t_1]$  if it is regulated by an affine open-loop controls of the form  $u(t) := u + u_1 t$ , the velocities of which are constant.*

Although we shall concentrate our study on inert evolutions, we shall provide some properties common to evolutions governed by open-loop controls  $u(t) := u + u_1 t + \dots + u_{m-1} \frac{t^{m-1}}{(m-1)!}$  which are  $(m - 1)$ -degree polynomials in time.

More generally, we are interested in evolutions governed by open-loop controls  $t \mapsto u(t)$  with bounded derivative  $u^{(m)}(t)$  for some  $m \geq 1$ .

In regulation systems, agents acting on state variables are well identified, but regulons evolve slowly by lack of a definite agent acting on controls or because the change of controls is costly, even very costly. The regulons are thus constrained by some inertia that can be estimated through some measure of their velocities. For instance, in economics, the parameters are prices or other fiduciary variables, and inflation should be bounded. We may even look for heavy evolutions when the regulons evolve as slowly as possible.

Naturally, passive evolutions may not exist, or exist only on a finite time interval. Then the question arises to study **when, where and how** passive evolutions must cease to be passive and allows the regulons to evolve in order to guarantee the viability: We shall give a name to this property which seems to be shared by so many systems dealing with living beings: In a loose way, the inertia principle states that the *“regulons” of the system are changed only when viability is at stake*. It runs against the teleological trend assigning aims to be achieved (in even an intertemporal optimal way) by the state of the system and the belief that actors control the system for such purposes.

The inertia principle stating that the *“regulons” of the system are changed only when viability is at stake* is a mathematical formulation of the concept of *punctuated equilibrium* introduced in paleontology by Eldredge and Gould in 1972. It runs against the teleological trend assigning aims to be achieved (in even an optimal way) by the state of the system and the belief that actors control the system for such purposes. However, they were anticipated by Darwin himself who added the sentence

*and lastly, although each species must have passed through numerous transitional stages, it is probable that the periods, during which each underwent modification, though many and long as measured by years, have been short in comparison with the periods during which each remained in an unchanged condition*

to Chapter XI of the sixth edition of *Origin of Species*.

The question arises how to change constant regulons when the viability condition is about to be violated. This can be done either brutally, by impulses or in a smoother manner, by minimizing some cost function on the velocity. Naturally, there are many other approaches between these two extremes to obey the inertial principle. The brutal changes are the topic of impulse control and/or hybrid systems (see for instance Aubin, 1999). The second one is the topic of this section.

## 4.1 Inertia Functions

An adequate way to handle concepts of evolutions governed by open-loop polynomial controls and differentiable open-loop controls is by means of inertia functions. They are value functions of optimal control problems where the criterion (i) is no longer an integral one, but involves instead the supremum

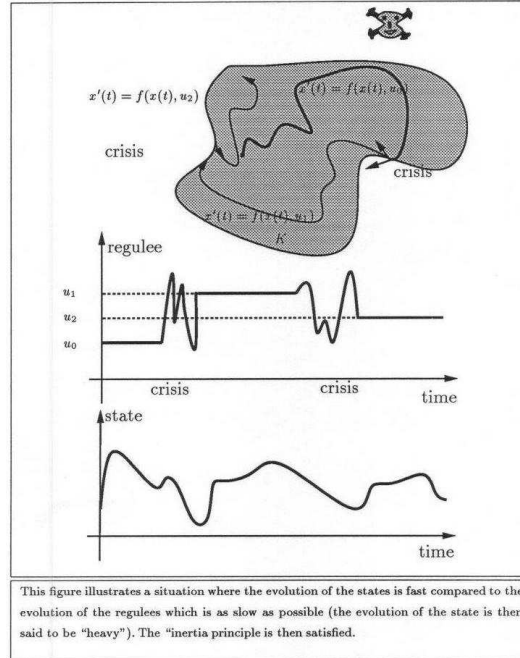


FIG. 68. – Inertia Principle and Heavy Solutions

Figure 1.4. *Punctuated Evolution.* Starting from  $x_0$  with the constant regulon  $u_0$ , the solution evolves in  $K$  until time  $t_1$ , (first *punctuated equilibrium phase*) when the state  $x(t_1)$  is about to leave  $K$  and when the constant regulon  $u_0$  must evolve. Then a *crisis* happens during which velocities also evolve (as slowly as possible) to maintain viability, until time  $\bar{t}_1$  where the control can remain constant during a nonempty time interval: *second punctuated equilibrium phase*.

over time of a norm (or, possibly, another criterion function); (ii) is acting on the  $m$ -th derivative of the control instead of the state-control pairs.

Let us introduce the  $\mathcal{P}_1(x, u) := \mathcal{P}_1(x, u)$  the set of solutions  $(x(\cdot), u(\cdot))$  to the above parameterized system (1.4) starting at  $(x, u)$ .

DEFINITION 4.3 *The first-order inertia function, or, in short, the inertia function  $\alpha$  of parameterized system (1.4) is the minimal worst intertemporal inertia  $\alpha_1(x, u)$  of the evolutions starting from  $(x, u) \in \text{Graph}(U)$  defined by*

$$\alpha_1(x, u) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{P}_1(x, u)} \sup_{t \geq 0} \|u'(t)\| \in [0, +\infty].$$

The zero-level set

$$\{(x, u) \in \text{Graph}(U) \text{ such that } \alpha_1(x, u) \leq 0\}$$

is called the *inertia set* of the system  $(f, U)$ . The set  $N(u) := N_1(u)$  of elements  $x \in \text{Dom}(U)$  such that  $\alpha_1(x, u) = 0$  as the (first-order) *viability niche*.

The domain of the inertia function  $\alpha_1$  is the subset of initial state-control pairs from which at least one evolution is governed by open-loop controls with bounded derivatives and its level sets

$$\{(x, u) \in \text{Graph}(U) \text{ such that } \alpha_1(x, u) \leq c\}$$

provide the subset of initial states from which at least one evolution  $(x(\cdot), u(\cdot))$  is governed by open-loop controls the derivatives of which are bounded by the given constant  $c$ .

We continue our investigation of the regulation by twice-differentiable open-loop controls, and, among them, affine controls of the form  $u(t) := u + u_1 t$  (the so-called *ramp controls*). For that purpose, we denote by  $\mathcal{P}_2(x, u, u_1)$  the set of solutions  $(x(\cdot), u(\cdot))$  to the control system (1.4) satisfying the initial conditions

$$x(0) = x, u(0) = u, u'(0) = u_1.$$

**DEFINITION 4.4** *The the second-order inertia function  $\alpha$  of parameterized system (1.4) is the minimal worst intertemporal inertia  $\alpha_2(x, u, u_1)$  defined by*

$$\alpha_2(x, u, u_1) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{P}_2(x, u, u_1)} \sup_{t \geq 0} \|u''(t)\| \in [0, +\infty].$$

We regard the zero-level set

$$\{(x, u, u_1) \text{ such that } \alpha_2(x, u, u_1) \leq 0\}$$

as the *second-order inertia set* and the set  $N_2(u, u_1)$  of elements  $x \in \text{Dom}(U)$  such that  $\alpha_2(x, u, u_1) = 0$  as the *second-order viability niche*.

We shall prove that

$$\alpha_2(x, u, u_1) = 0 \text{ if and only if } \forall t \geq 0, \alpha_1(x(t), u + u_1 t) = \|u_1\|.$$

Generally, we associate with any integer  $m \geq 0$  the set  $\mathcal{P}_m(x, u, \dots, u_{m-1})$  of solutions  $(x(\cdot), u(\cdot))$  to the control system (1.4) satisfying the initial conditions

$$x(0) = x, u(0) = u, u'(0) = u_1, \dots, u^{m-1}(0) = u_{m-1}.$$

**DEFINITION 4.5** *The  $m$ -order inertia function  $\alpha_m$  of parameterized system (1.4) is the minimal worst intertemporal inertia  $\alpha_m(x, u, \dots, u_{m-1})$  of the evolutions starting from  $(x, u, \dots, u_{m-1})$  defined by*

$$\alpha_m(x, u, \dots, u_{m-1}) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{P}_m(x, u, \dots, u_{m-1})} \sup_{t \geq 0} \|u^{(m)}(t)\| \in [0, +\infty].$$

We regard the zero-level set

$$\{(x, u, \dots, u_{m-1}) \text{ such that } \alpha_m(x, u, \dots, u_{m-1}) \leq 0\}$$

as the  $m$ -order inertia set and the set  $N_m(u, u_1, \dots, u_{m-1})$  of elements  $x \in \text{Dom}(U)$  such that  $\alpha_m(x, u, u_1, \dots, u_{m-1}) = 0$  as the  $m$ -order viability niche.

In other words, from any  $x$  belonging to a  $m$ -order viability niche  $N_m(u, u_1, \dots, u_{m-1})$  starts at least an evolution governed by an open-loop control  $u(t) := u + u_1 t + \dots + u_{m-1} \frac{t^{m-1}}{(m-1)!}$  which is a  $(m-1)$ -degree polynomial in time.

For characterizing the inertia function in terms of viability kernels, we introduce the following auxiliary  $m$ -metasystem:

$$\left\{ \begin{array}{ll} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u'(t) = u_1(t) \\ \dots & \dots \\ (m+1) & u'_{m-1}(t) = u_m(t) \\ (m+2) & y'(t) = 0 \\ & \text{where } \|u_m(t)\| \leq y(t) \end{array} \right. \quad (1.5)$$

of differential inclusions subjected to the constraint

$$\forall t \geq 0, (x(t), u(t), \dots, u_{m-1}(t), y(t)) \in \text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+.$$

**THEOREM 4.6** *For any  $m \geq 1$ , the  $m$ -order inertia function is related to the viability kernel of  $\text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+$  under auxiliary metasystem (1.5) by the formula*

$$\alpha_m(x, u, \dots, u_{m-1}) = \inf_{(x, u, \dots, u_{m-1}, y) \in \text{Viab}_{(1.5)}(\text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+)} y.$$

*If  $f$  is continuous and the graph of  $U$  is closed, then, from any  $(x, u, \dots, u_{m-1}) \in \text{Dom}(\alpha_m)$  starts at least one evolution  $(x(\cdot), u(\cdot)) \in \mathcal{P}_m(x, u, \dots, u_{m-1})$  such that*

$$\alpha_m(x, u, \dots, u_{m-1}) := \sup_{t \geq 0} \|u^{(m)}(t)\|.$$

*In particular, from any  $x$  belonging to a  $m$ -order viability niche  $N_m(u, u_1, \dots, u_{m-1})$  starts at least an evolution governed by an open-loop control  $u(t) = u + u_1 t + \dots + u_{m-1} \frac{t^{m-1}}{(m-1)!}$  which is a  $(m-1)$ -degree polynomial in time.*

**Proof** — Indeed, to say that  $(x, u, \dots, u_{m-1}, y)$  belongs to  $\text{Viab}_{(1.5)}(\text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+)$  amounts to saying that there exists an evolution  $t \mapsto (x(t), u(t), u_1(t), \dots, u_m(t), y(t))$  governed by control system (1.4) such that  $y(t) = t$  and  $u_j(t) = u^{(j)}(t)$ , and thus a solution  $(x(\cdot), u(\cdot)) \in \mathcal{P}_m(x, u, \dots, u_{m-1})$  satisfying

$$\forall t \geq 0, \|u^{(m)}(t)\| \leq y$$

so that  $\alpha_m(x, u, \dots, u_{m-1}) \leq y$ .

Conversely, we can associate with any  $\varepsilon > 0$  an evolution  $(x_\varepsilon(\cdot), u_\varepsilon(\cdot))$  belonging to  $\mathcal{P}_m(x, u, \dots, u_{m-1})$  such that

$$\forall t \geq 0, \|u_\varepsilon^m(t)\| \leq \alpha(x, u, \dots, u_{m-1}) + \varepsilon =: y_\varepsilon.$$

Therefore, setting  $u_{\varepsilon_j}(t) := u_{\varepsilon}^j(t)$  and  $y_{\varepsilon}(t) = y_{\varepsilon}$ , we observe that  $t \mapsto (x_{\varepsilon}(t), u_{\varepsilon}(t), u_{\varepsilon_1}(t), u_{\varepsilon_{m-1}}(t), y_{\varepsilon})(t)$  is a solution to the auxiliary system (1.5) viable in  $\text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+$ , and thus, that  $(x, u, \dots, u_{m-1}, y_{\varepsilon})$  belongs to  $\text{Viab}_{(1.5)}(\text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+)$ . Therefore

$$(x, u, \dots, u_{m-1}, y) \in \text{Viab}_{(1.5)}(\text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+) \quad y \leq y_{\varepsilon} := \alpha(x, u, \dots, u_{m-1}) + \varepsilon$$

and it is enough to let  $\varepsilon$  converge to 0.

Since the auxiliary system (1.5) is Marchaud whenever  $m \geq 1$  and  $f$  continuous and since the auxiliary constrained set  $\text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+$  is closed by assumption, then the viability kernel  $\text{Viab}_{(1.5)}(\text{Graph}(U) \times X^{m-1} \times \mathbb{R}_+)$  is also closed and the upper semi-compactness of the associated evolutionary system implies that there exists a subsequence (again denoted by) of  $(x_{\varepsilon}(\cdot), u_{\varepsilon_0}(\cdot), u_{\varepsilon_{m-1}}(\cdot), y_{\varepsilon})(\cdot)$  which converge to a solution  $(x(\cdot), u(\cdot), \dots, u_{m-1}(\cdot), \alpha(x, u, \dots, u_{m-1}))$  satisfying

$$\forall t \geq 0, \quad \|u^m(t)\| \leq \alpha(x, u, \dots, u_{m-1}).$$

Therefore, the infimum of the  $m$ -order inertia function is achieved. In particular, when  $\alpha(x, u, \dots, u_{m-1}) = 0$ , we infer that for all  $t \geq 0$ ,  $\|u^m(t)\| = 0$  and thus, that  $u(t) = u + u_1 t + \dots + u_{m-1} \frac{t^{m-1}}{(m-1)!}$ . ■

The  $m$ -order viability niches “lock-in” evolutions which enter them in the following sense:

**PROPOSITION 4.7** *If at some time  $t_f$ ,  $x(t_f)$  belongs to the  $m$ -order viability niche  $N_m(u(t_f), u'(t_f), \dots, u^{(m-1)}(t_f))$ , then for  $t \geq t_f$ , the evolution  $x(\cdot) \in \mathcal{S}(x)$  may be regulated by the open-loop polynomial  $u(t) = u(t_f) + u'(t_f)t + \dots + u^{(m-1)}(t_f) \frac{t^{m-1}}{(m-1)!}$  and remain in this viability niche forever.*

**Remark: Hamilton-Jacobi-Bellman Equations** One can prove that the **inertia function**  $\alpha_m$  is the smallest nonnegative solution to the Hamilton-Jacobi-Bellman partial differential equation

$$\forall (x, u) \in \text{Graph}(U), \quad \sum_{i=1}^n \frac{\partial \mathbf{v}(x, u)}{\partial x_i} f_i(x, u) - \mathbf{v}(x, u) \left\| \frac{\partial \mathbf{v}(x, u)}{\partial u} \right\| = 0$$

for  $m = 1$  and, for  $m \geq 2$ ,

$$\begin{cases} \frac{\partial \mathbf{v}(x, u, u_1, \dots, u_{m-1})}{\partial x} f(x, u) + \sum_{j=0}^{m-2} \frac{\partial \mathbf{v}(x, u, u_1, \dots, u_{m-1})}{\partial u_j} u_{j+1} \\ - \mathbf{v}(x, u, u_1, \dots, u_{m-1}) \left\| \frac{\partial \mathbf{v}(x, u, u_1, \dots, u_{m-1})}{\partial u_{m-1}} \right\| = 0. \end{cases}$$

## 4.2 Metasystems

The level-sets of the inertia function can be obtained as viability kernels of “meta-systems” which we now describe in the case of first-order inertia functions only. Indeed, we ask the model to satisfy an **inertia constraint**: The norm of the velocity of the regulon (or any cost function on the velocity of the regulon) must be bounded by a constant  $c$ . Requiring that the norm of the velocity of the control  $\|u'(t)\|$  is bounded by a constant  $c$  amounts to writing that for all  $t \geq 0$ ,  $u'(t)$  belongs to the ball  $B(0, c)$  centered in 0 and of radius  $c$ .

This is just a (simple) differential inclusion  $u'(t) \in B(0, c)$  that can be added to the initial differential equation (1.4)(i) to form a more balanced system

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u'(t) \in B(0, c) \end{cases} \quad (1.6)$$

of differential inclusions (the meta-system). The output-input regulation  $u(t) \in U(x(t))$  of (1.4) becomes a constrained set  $\{(x(t), u(t)) \in \text{Graph}(U)\}$  of the new system.

However, there is no reason that for any initial pair  $(x, u) \in \text{Graph}(U)$ , there should exist a solution  $(x(\cdot), u(\cdot))$  of system (1.6) such that  $x(\cdot)$  is an evolution of system (1.4), that is a solution satisfying the set-valued feedback constraint  $u(t) \in U(x(t))$ , i.e., the viability constraint

$$\forall t \geq 0, (x(t), u(t)) \in \text{Graph}(U).$$

The set of such initial states is the viability kernel of  $\text{Graph}(U)$  under system (1.6).

Since the viability kernel  $\text{Viab}(\text{Graph}(U))$  is a subset of  $X \times \mathcal{U}$ , it can be regarded as the graph of a set-valued map  $U_c$  defined by

$$\text{Graph}(U_c) := \text{Viab}_{(1.6)}(\text{Graph}(U)). \quad (1.7)$$

DEFINITION 4.8 The *metasystem* associated with the parameterized system (1.4)

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in U(x(t)) \end{cases}$$

and a velocity bound  $c > 0$  is the system (1.6)

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & \|u'(t)\| \leq c. \end{cases}$$

The set-valued map  $U_c : X \rightsquigarrow \mathcal{U}$  defined by (1.7)

$$\text{Graph}(U_c) := \text{Viab}_{(1.6)}(\text{Graph}(U))$$

is called the *c-regulation map*.

The pairs  $(x, u)$  are called the *metastates* and the derivatives  $v(t) := u'(t)$  of the *regulons* are called the *metaparameters* (or *metacontrols*, *metaregulons*, etc.).

Starting from an initial metastate  $(x_0, u_0) \in \text{Graph}(U_c)$ , there exists a *continuous* evolution  $(x(\cdot), u(\cdot)) \in \mathcal{P}(x, u)$  satisfying the further condition

$$\forall t \geq 0, u(t) \in U_c(x(t)) \subset U(x(t)).$$

Therefore,  $U_c$  provides a regulation map, the selections of which are (static) feedbacks governing the evolution of continuous state-regulon pairs. The Viability Theorem provides a regulation map governing the evolution of the velocities

of the controls, the selections of which are dynamical feedbacks. In economics, dynamic retroactions are adjustment laws (that are often given *a priori*, as the supply and demand law, the Walrasian tâtonnement, etc.  $c$ -Regulation maps can be studied thanks to the inertia function:

**PROPOSITION 4.9** *If  $f$  is continuous and the graph of  $U$  is closed, the  $c$ -regulation map  $U_c$  of system (1.6) defined by (1.7) is related to the inertia function by the formula*

$$U_c(x) = \{u \in U(x) \text{ such that } \alpha(x, u) \leq c\}.$$

*The level sets  $\{(x, u) \mid \alpha(x, u) \leq c\}$  of the inertia function are viable under the metasytem (1.6). In particular, a  $c$ -level sets of the inertia function is viable under  $x' = f(x, \tilde{u}(x))$  if and only if the feedback  $\tilde{u}$  is a selection of  $U_c$ :*

$$\forall x \in [a, b], \tilde{u}(x) \in U_c(x).$$

**Proof** — The inclusion

$$U_c(x) \subset \{u \in U(x) \text{ such that } \alpha(x, u) \leq c\}$$

is always true: If  $(x, u)$  belongs to the graph of  $U_c$ , which is the viability kernel of the graph of  $U$  under the metasytem (1.6), there exists an evolution  $(x(\cdot), u(\cdot))$  such that, at each instant  $t \geq 0$ ,  $x'(t) = f(x(t), u(t))$ ,  $u(t) \in U_c(x(t)) \subset U(x(t))$  and  $\alpha(x, u) \leq \|u'(t)\| \leq c$ . The converse is true because there exists an inert evolution  $(x(\cdot), u(\cdot))$  such that  $\|u'(t)\| = \alpha(x, u) \leq c$  thanks to Theorem 4.6. ■

### 4.3 Implementing the Inertia Principle by Heavy Evolutions

How can we implement the inertia principle? For instance, one can select at each instant the regulons providing viable evolutions with *minimal velocity*. This is an example that obeys this inertia principle. Evolutions obtained in this way are called “heavy” viable evolutions<sup>10</sup> in the sense of heavy trends in economics. To define heavy solutions, we still fix a bound  $c$  on the norms of the velocities of the regulons and take any initial metastate  $(x, u)$  such that  $\alpha(x, u) < c$ . We then fix the regulon  $u$  and consider the passive<sup>11</sup> evolution  $(x_u(t), u)$  where  $x_u(\cdot)$  is the solution to differential equation  $x'(t) = f(x(t), u)$  (evolving with velocity  $u'(t) = 0$ ).

As long as  $\alpha(x_u(t), u)$  is smaller than the velocity bound  $c$ , the regulon  $u$  inherited from the past can be maintained, allowing the system to obey the inertia principle. Since the state  $x_u(\cdot)$  of the system evolves while the regulon

<sup>10</sup>When the regulons are the velocities, heavy solutions are the ones with minimal acceleration, i.e., maximal inertia.

<sup>11</sup>We assume here for simplicity that the solutions to the differential equations  $x'(t) = f(x(t), u)$  are unique. This is the case when the maps  $x \mapsto f(x, u)$  are Lipschitz or monotone.

remains constant and equal to  $u$ , the inertia function  $\alpha(x_u(t), u)$  evaluated on such an evolution, equal to  $\alpha(x, u) < c$  at initial time  $t = 0$ , measures the intertemporal cost of changing regulons. It may increase and eventually overrun the bound  $c$  measuring the maximal velocity of the regulons at some time  $\tau_c(x, u)$  at the state  $\xi_c(x, u) := x_u(\tau_c(x, u))$  providing warning signals:

**DEFINITION 4.10** *Assume that  $c > \alpha(x, u)$ . Then the warning time  $\tau_c(x, u) \in \mathbb{R} \cup \{+\infty\}$  is the first instant when evolutions  $x_u(\cdot)$  starting from  $x$  leaves  $U_c^{-1}(u)$  at the warning state  $\xi_c(x, u) := x_u(\tau_c(x, u)) \in \Xi_c(u)$  where  $\Xi_c$  is the signal map associating with any regulon  $u$  the set*

$$\Xi_c(u) := \{x \in \text{Dom}(U) \text{ such that } \alpha(x, u) = c.\}$$

Warning signals tell us **when, where and how the regulons must evolve**, defining a period of viability crisis: **when** is given by  $\tau_c(x, u)$  and **where** corresponds to  $\xi_c(x, u)$ . To survive, other regulons must operate such that the new velocity drives the evolution to respect the velocity limit and/or to remain inside the graph of  $U$  until the regulon can again remain constant for a new period of time.

#### 4.4 The Transition Cost Function

The inertia function defined in Definition 4.3 offers the simplest example of cost incurred by changing regulons. The general form of costs incurred by changing regulons is given by:

**DEFINITION 4.11** *Consider a nonnegative extended cost function  $\mathbf{c} : X \times \mathcal{U} \times \mathcal{U} \mapsto \mathbb{R}_+$ , a cumulated transition cost function of the regulons  $\mathbf{l} : X \times \mathcal{U} \times \mathcal{U} \rightsquigarrow \mathbb{R}_+ \cup \{+\infty\}$  and a discount factor  $m(x, u)$  (which may depend upon both of the states and the controls). The transition function  $\alpha_{(\mathbf{c}, \mathbf{l})}$  is defined by*

$$\left\{ \begin{array}{l} \alpha_{(\mathbf{c}, \mathbf{l})}(x, u) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{P}(x, u)} \sup_{t \geq 0} \left( e^{-\int_0^t m(x(s), u(s)) ds} \mathbf{c}(x(t), u(t), u'(t)) \sigma \right. \\ \left. + \int_0^t e^{-\int_0^\tau m(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau), u'(\tau)) d\tau \right). \end{array} \right. \quad (1.8)$$

Starting from an initial stat  $x$ , it will be advantageous to look for an initial regulon  $u \in U(x)$  that minimizes the worst transition cost of regulons. To characterize the transition cost function in terms of viability kernels, we introduce the set-valued map

$$S_{\mathbf{c}} : (x, u; y) \rightsquigarrow \{v \in \mathcal{U} \mid \mathbf{c}(x, u, v) \leq y\}$$

and the following auxiliary metasystem of differential inclusions

$$\left\{ \begin{array}{l} (i) \quad x'(t) = f(x(t), u(t)) \\ (ii) \quad u'(t) = v(t) \\ (iii) \quad y'(t) = m(x(t), u(t))y(t) - \mathbf{l}(x(t), u(t), v(t)) \\ \text{where } v(t) \in S_{\mathbf{c}}(x(t), u(t); y(t)) \end{array} \right. \quad (1.9)$$

subject to the constraint

$$\forall t \geq 0, (x(t), u(t), y(t)) \in \text{Graph}(U) \times \mathbb{R}_+.$$

**THEOREM 4.12** *The transition cost function is related to the viability kernel of the graph of  $U$  under the auxiliary metasystem (1.9) by the following formula*

$$\alpha_{(\mathbf{c}, \mathbf{l})}(x, u) = \inf_{(x, u, y) \in \text{Viab}_{(1.9)}(\text{Graph}(U) \times \mathbb{R})} y.$$

**Proof** — Indeed, to say that  $(x, u, y)$  belongs to the viability kernel of the graph of  $U$  under the auxiliary system (1.9) amounts to saying that there exists an evolution  $t \mapsto (x(t), u(t), y(t))$  governed by the auxiliary metasystem such that, for all  $t \geq 0$ ,  $u(t) \in U(x(t))$ . By definition of (1.9), we know that for all  $t \geq 0$ , this evolution satisfies also for all  $t \geq 0$ ,

$$\mathbf{c}(x(t), u(t), v(t)) \leq y(t)$$

where

$$y(t) = e^{\int_0^t m(x(s), u(s)) ds} \left( y - \int_0^t e^{-\int_0^\tau m(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau), v(\tau)) d\tau \right).$$

Therefore

$$\sup_{t \geq 0} \left( e^{-\int_0^t m(x(s), u(s)) ds} \mathbf{c}(x(t), u(t), u'(t)) + \int_0^t e^{-\int_0^\tau m(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau), u'(\tau)) d\tau \right) \leq y.$$

and thus,  $\alpha_{(\mathbf{c}, \mathbf{l})}(x, u) \leq \inf_{(x, u, y) \in \text{Viab}_{(1.9)}(\text{Graph}(U) \times \mathbb{R}_+)} y$ .

Conversely, we know that for any  $\varepsilon > 0$ , there exists an evolution  $(x(\cdot), u(\cdot)) \in \mathcal{P}(x, u)$  such that

$$\sup_{t \geq 0} \left( e^{-\int_0^t m(x(s), u(s)) ds} \mathbf{c}(x(t), u(t), u'(t)) + \int_0^t e^{-\int_0^\tau m(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau), u'(\tau)) d\tau \right) \leq \alpha_{(\mathbf{c}, \mathbf{l})}(x, u) + \varepsilon.$$

Setting

$$y_\varepsilon(t) := e^{\int_0^t m(x(s), u(s)) ds} \left( \alpha_{(\mathbf{c}, \mathbf{l})}(x, u) + \varepsilon - \int_0^t e^{-\int_0^\tau m(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau), u'(\tau)) d\tau \right)$$

we infer that  $\mathbf{c}(x(t), u(t), v(t)) \leq y_\varepsilon(t)$  and thus, that  $t \mapsto (x(t), u(t), y_\varepsilon(t))$  is a solution to the solution to auxiliary evolutionary system (1.9) starting at  $(x, u, \alpha_{(\mathbf{c}, \mathbf{l})}(x, u) + \varepsilon)$ . This evolution is viable in  $\text{Graph}(U) \times \mathbb{R}_+$  since  $(x(\cdot), u(\cdot)) \in \mathcal{P}(x, u)$ , and thus, since  $x(t) \in U(x(t))$ , or, equivalently, since

$$\forall t \geq 0, ((x(t), u(t), y_\varepsilon(t)) \in \text{Graph}(U) \times \mathbb{R}.$$

Hence  $(x, u, \alpha_{(\mathbf{c}, \mathbf{l})}(x, u) + \varepsilon)$  belongs to the viability kernel  $\text{Viab}_{(1.9)}(\text{Graph}(U) \times \mathbb{R}_+)$ , so that

$$\inf_{(x, u, y) \in \text{Viab}_{(1.9)}(\text{Graph}(U) \times \mathbb{R}_+)} y \leq \alpha_{(\mathbf{c}, \mathbf{l})}(x, u) + \varepsilon.$$

Letting  $\varepsilon$  converge to 0, we obtain the converse inequality. ■

## 5. Infinite Horizon Intergenerational Optimization

We refer to Haurie, 2003 for the relevance of infinite horizon control problems in climate policy assessment. We just propose here a complement to this paper. Indeed, the concept of viability kernel is not only interesting by itself, but it happens to be a mathematical and numerical tool for solving other mathematical problems. In particular, following the papers of H. Frankowska<sup>12</sup>, we can characterize the epigraph (see for instance Aubin, 1998) of the value function of infinite horizon optimal control problems as a viability kernel (epigraphical approach of the Hamilton-Jacobi-Bellman approach). We shall bypass the Hamilton-Jacobi-Bellman partial differential equation, since we can compute the value function directly by the Saint-Pierre Viability Kernel Algorithm (and obtain the optimal trajectories). The perverse irony of all that is that viability theory which was conceived in the end of the 1970's as a mathematical metaphor of Darwinian evolution designed to replace intertemporal optimization provided mathematical tools to solve optimal control problems via the Hamilton-Jacobi-Bellman viewpoint, as advocated by H el ene Frankowska.

The usual shape of a criterion on the space  $\mathcal{C}(0, \infty; X)$  of continuous functions involves a ‘‘transient state’’ cost function (often called a Lagrangian)  $\mathbf{l} : X \times \mathcal{U} \mapsto \mathbb{R}_+$ . We introduce an ‘‘intergenerational cost function’’  $\mathbf{d} : X \mapsto \mathbb{R}_+ \cup \{+\infty\}$  and denote by  $\mathcal{D}(x)$  the subset of evolutions  $(x(\cdot), u(\cdot)) \in \mathcal{P}(x)$  satisfying the intergenerational constraints

$$\forall t \geq 0, \int_t^\infty e^{-m\tau} \mathbf{l}(x(\tau), u(\tau)) d\tau \leq \mathbf{d}(x(t)). \quad (1.10)$$

This expresses that, at each instant  $t$ , the *future cumulated cost*

$$\int_t^\infty e^{-m\tau} \mathbf{l}(x(\tau), u(\tau)) d\tau$$

should remain below a given cost  $\mathbf{d}(x(t))$  of the value of the state at time  $t$ .

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<sup>12</sup>The epigraph of a real-valued function is what is above its graph. The discovery that the main properties of a function involved and used in optimization problems deal with the epigraph of a function took its roots in the sixties in convex analysis after the pioneering works of Moreau and Rockafellar. The viability of the epigraph of a function was used since 1981 in the context of Lyapunov functions. Their more and more frequent use in optimization and control since the sixties is handy because it allows to replace inequalities constraints by membership relations, and becomes more and more familiar. It was taken up by H el ene Frankowska for characterizing value functions of optimal control problems in a long series of papers. H el ene Frankowska proved that the epigraph of the value function of an optimal control problem — assumed to be only lower semicontinuous — is semi-permeable, (i.e., invariant and backward viable) under a (natural) auxiliary system. Furthermore, when it is continuous, she proved that its epigraph is viable and its hypograph invariant (Frankowska, 1989; Frankowska, 1989b; Frankowska, 1993). By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton-Jacobi equation in the sense of M. Crandall and P.-L. Lions. This epigraphical approach in the field of Hamilton-Jacobi equations has since been taken up by other authors.

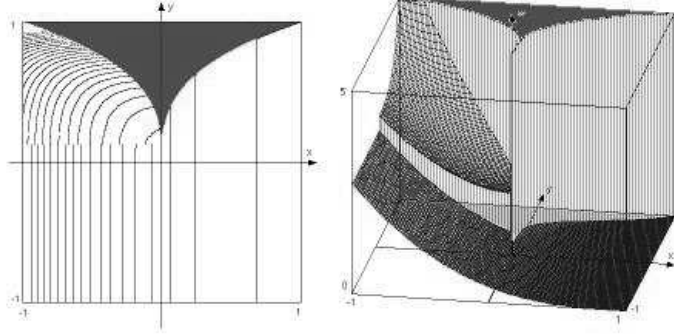


Figure 1.5. Example of the Value Function of an Infinite Control Problem where the dynamic is:  $(x'_1, x'_2) = (u, 0)$ , with  $u \in \mathcal{U} = [-1, 1]$ : Minimize  $\int_0^{+\infty} (x_1(s) - 1)^2 e^{-\lambda s} ds$  under the constraint  $K = \{(x_1, x_2) \mid -1 \leq x_2 \leq |x_1|^{\frac{1}{3}}, -1 \leq x_1 \leq 1\}$ . Source: Patrick Saint-Pierre.

DEFINITION 5.1 *The intergenerational valuation function  $\tilde{V}_{\mathbf{d}}^{\infty}$  of the infinite horizon intertemporal minimization problem*

$$\tilde{V}_{\mathbf{d}}^{\infty}(x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{D}(x)} \int_0^{\infty} e^{-m\tau} \mathbf{l}(x(\tau), u(\tau)) d\tau$$

over all the evolutions satisfying the intergenerational constraints (1.10)

$$\forall t \geq 0, \int_t^{\infty} e^{-m\tau} \mathbf{l}(x(\tau), u(\tau)) d\tau \leq \mathbf{d}(x(t))$$

is called the intergenerational valuation function.

When  $\mathbf{d} \equiv +\infty$ , the intergenerational constraints disappear, and  $\tilde{V}_{\infty}^{\infty}$  is the standard value function of an infinite horizon optimal control problem.

Let us observe that if  $x(\cdot)$  satisfies the intergenerational constraints, then

$$0 \leq \int_0^{\infty} e^{-m\tau} \mathbf{l}(x(\tau), u(\tau)) d\tau \leq \mathbf{d}(x)$$

so that

$$0 \leq \tilde{V}_{\mathbf{d}}^{\infty}(x) \leq \mathbf{d}(x).$$

Hence, whenever  $\mathbf{d}(x)$  is finite, we infer that

$$\begin{cases} \forall t \geq 0, \int_0^{\infty} e^{-m\tau} \mathbf{l}(x(\tau), u(\tau)) d\tau \\ \leq e^{-mt} \mathbf{d}(x(t)) + \int_0^t e^{-m\tau} \mathbf{l}(x(\tau), u(\tau)) d\tau. \end{cases}$$

Let us introduce the auxiliary control system

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & y'(t) = my(t) - \mathbf{l}(x(t), u(t)) \\ & \text{where } u(t) \in U(x(t)). \end{cases} \quad (1.11)$$

**THEOREM 5.2** *Assume that the extended function  $\mathbf{d}$  is nontrivial and non negative and that the Lagrangian  $\mathbf{l}$  is non negative. Consider the viability kernel  $\text{Viab}_{(1.11)}(\mathcal{K})$  of the subset*

$$\mathcal{K} := \{(x, y) \in X \times \mathbf{R}_+ \mid y \leq \mathbf{d}(x)\}$$

*under auxiliary the set-valued evolutionary system (1.11). Then*

$$\forall x \in \mathcal{K}, \bar{V}_{\mathbf{d}}^{\infty}(x) = \inf\{y \mid (x, y) \in \text{Viab}_{(1.11)}(\mathcal{K})\}.$$

Being a viability kernel, the epigraph of infinite horizon optimal control problems can be computed by the Viability Kernel Algorithm (see Figure 1.5 for an example of valuation function of an infinite horizon optimal control with state constraints).

## 6. Characterization of Viability and/or Capturability

The main task is to characterize the subsets having this viability/capturability property. To be of value, this task must be done without solving the system for checking the existence of viable solutions for each initial state.

### 6.1 Tangent Directions

An immediate intuitive idea jumps to the mind: at each point on the boundary of the constrained set outside the target, where the viability of the system is at stake, there should exist a velocity which is in some sense **tangent** to the viability domain and serves to allow the solution to bounce back and remain inside it. This is, in essence, what the viability theorem below states. Before stating it, the mathematical implementation of the concept of tangency must be made.

We cannot be content with viability sets that are smooth manifolds (such as spheres, which have no interior), because inequality constraints would thereby be ruled out (as for balls, that possess distinct boundaries). So, we need to implement” the concept of a direction  $v$  tangent to  $K$  at  $x \in K$ , which should mean that starting from  $x$  in the direction  $v$ , we do not go too far from  $K$ : The adequate definition due to G. Bouligand and F. Severi proposed in 1930 states that a direction  $v$  is **tangent to  $K$  at  $x \in K$**  if it is a limit of a sequence of directions  $v_n$  such that  $x + h_n v_n$  belongs to  $K$  for some sequence  $h_n \rightarrow 0+$ . The collection of such directions, which are in some sense inward”, constitutes

a closed cone  $T_K(x)$ , called the **tangent cone**<sup>13</sup> to  $K$  at  $x$ . Naturally, except if  $K$  is a smooth manifold, we lose the fact that the set of tangent vectors is a vector-space, but this discomfort is not unbearable, since advances in set-valued analysis built a calculus of these cones allowing us to compute them. See Aubin & Frankowska, 1990 and Rockfellar & Wets, 1997 for instance.

## 6.2 The Adaptive Map

We then associate with the dynamical system (described by  $(f, U)$ ) and with the viability constraints (described by  $K$ ) the (set-valued) adaptive or regulation map  $R_K$ . It maps any state  $x \in K \setminus C$  to the subset  $R_K(x)$  (possibly empty) consisting of regulons  $u \in U(x)$  which are viable in the sense that  $f(x, u)$  is tangent to  $K$  at  $x$ :

$$R_K(x) := \{u \in U(x) \mid f(x, u) \in T_K(x)\}.$$

We can for instance *compute the* adaptive map in many instances.

## 6.3 The Viability Theorem

The Viability Theorem states that *the target  $C$  can be reached in finite time from each initial condition  $x \in K \setminus C$  by at least one evolution of the control system viable in  $K$  if and only if for every  $x \in K \setminus C$ , there exists at least one viable control  $u \in R_K(x)$ .*

This Viability Theorem holds true when both  $C$  and  $K$  are closed and for a rather large class of systems, called **Marchaud systems**: Beyond imposing some weak technical conditions, the only severe restriction is that, for each state  $x$ , the set of velocities  $f(x, u)$  when  $u$  ranges over  $U(x)$  is **convex** (This happens for the class of control systems of the form

$$x'(t) = f(x(t)) + G(x(t))u(t)$$

where  $G(x)$  are linear operators from the control space to the state space, when the maps  $f : X \mapsto X$  and  $G : X \mapsto \mathcal{L}(Y, X)$  are continuous and when the control set  $U$  (or the images  $U(x)$ ) are convex).

Curiously enough, viability implies stationarity, i.e., the existence of an equilibrium. Equilibria being specific evolutions, their existence requires stronger assumptions. The Equilibrium Theorem states that *when the constrained set is assumed to be viable, convex and compact, then there exists a (viable) equilibrium*. Without convexity, we deduce only the existence of minimal viable closed subsets.

The proofs of the above Viability Theorem and the Equilibrium Theorem are difficult: The Equilibrium Theorem is derived from the 1910 Brouwer Fixed Point Theorem, and the proof of the Viability Theorem uses all the theorems

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<sup>13</sup>replacing the linear structure underlying the use of tangent spaces by the tangent cone is at the root of Set-Valued Analysis.

of functional analysis except the closed graph theorem and the Lebesgue Convergence Theorem. However, their consequences are much easier to obtain and can be handled with moderate mathematical competence.

## 6.4 The Adaptation Law

Once this is done, and whenever a constrained subset is viable for a control system, the second task is to show how to govern the evolution of viable evolutions. We thus prove that viable evolution of system (1.3) are governed by

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in R_K(x(t)) \text{ (adaptation law)} \end{cases} \quad (1.12)$$

until the state reaches the target  $C$ .

We observe that the initial set-valued map  $U$  involved in (1.3)ii) is replaced by the adaptive map  $R_K$  in (1.12)ii). The inclusion  $u(t) \in R_K(x(t))$  can be regarded as an adaptation law (rather than a learning law, since there is no storage of information at this stage of modelling).

## 6.5 Planning Tasks: Qualitative Dynamics

Reaching a target is not enough for studying the behavior of control systems, that have to plan tasks in a given order. This issue has been recently revisited in Aubin & Dordan, 2001 in the framework of qualitative physics (see Dordan, 1995), Eisenack & Petschel-Held 2002 and Aubin, 1996 for more details on this topic). We describe the sequence of tasks or objectives by a family of subsets regarded as qualitative cells. Giving an order of visit of these cells, the problem is to find an evolution visiting these cells in the prescribed order.

## 6.6 Static Viable Feedbacks

A (static) feedback  $r$  is a map  $x \in K \mapsto r(x) \in X$  which is used to pilot evolutions governed by the differential equation  $x'(t) = f(x(t), r(x(t)))$ . A feedback  $r$  is said to be viable if the solutions to the differential equation  $x' = f(x, r(x))$  are viable in  $K$ . The most celebrated examples of linear feedbacks in linear control theory designed to control a system have no reason to be viable for an arbitrary constrained set  $K$ , and, according to the constrained set  $K$ , the viable feedbacks are not necessarily linear.

However, the Viability Theorem implies that a feedback  $r$  is viable if and only if  $r$  is a selection of the adaptive map  $R_K$  in the sense that

$$\forall x \in K \setminus C, \quad r(x) \in R_K(x). \quad (1.13)$$

Hence, the method for designing feedbacks for control systems to evolve in a constrained subset amounts to find selections  $r(x)$ . One can design a factory" for designing selections (see Chapter 6 of Aubin, 1991, for instance). Ideally, a feedback should be continuous to guarantee the existence of a solution to the differential equation  $x' = f(x, r(x))$ . But this is not always possible. This is

the case of slow selection  $r^\circ$  of  $R_K$  of minimal norm, governing the evolution of slow viable evolutions (despite its lack of continuity).

## 6.7 Restoring Viability

There are no reasons why an arbitrary subset  $K$  should be viable under a control system. Therefore, the problem of reestablishing viability arises. One can imagine several methods for this purpose:

- 1 keep the constraints and change initial dynamics by introducing regulons that are viability multipliers”;
- 2 change the initial conditions by introducing a **reset map**  $\Phi$  mapping any state of  $K$  to a (possibly empty) set  $\Phi(x) \subset X$  of new initialized states” (impulse control), as in Aubin, 1999;
- 3 keep the same dynamics and looking for viable constrained subsets by letting the set of constraints evolve according to mutational equations, as in Aubin, 1999.

## 7. Conclusion

The purpose of viability theory is to attempt to answer the problem of adaptation to the environment. This is the case in biology, since the Claude Bernard’s “constance du milieu intérieur” and the ”homeostasis” of Walter Cannon. This is naturally the case in ecology and environmental studies. This is also the case in economics when we have to adapt to scarcity constraints, balances between supply and demand, and many other ones. In this paper we have tried, within this context, to answer different questions arising in climate change models when coupling with economic interaction.

Viability theory is quite appropriated to study dynamical systems where the environment is described by constraints of various kinds (representing objectives, physical and economic constraints, ”stability” constraints, etc.) that can never be violated. In the same time, the actions, the messages, the coalitions of actors and connectionist operators do evolve, and their evolution must be consistent with the constraints, with objectives reached at (successive) finite times (and/or must be selected through intertemporal criteria). We must not forget that there is no reason why collective constraints should be satisfied at each instant by evolutions under uncertainty governed by stochastic or tyochastic control systems. So this approach leads to the study of how actors may correct either the dynamics, and/or the constraints in order to reestablish this consistency. This also may allow us to provide an explanation of the formation and the evolution of the architecture of the system and of their variables, especially when considering large time scales.

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