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Dynamical Connectionist Network and Cooperative Games

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Jean-Pierre Aubin

14 rue Domat, F-75005 Paris,France
J.P.Aubin@wanadoo.fr
http://www.crea.polytechnique.fr/personnels/fiches/aubin/cvjpa.html

Abstract

Socio-economic networks, neural networks and genetic networks describe collective phenomena through constraints relating actions of several players, coalitions of these players and multilinear connectionist operators acting on the set of actions of each coalition. Static and dynamical cooperative games also involve coalitions. Allowing "coalitions to evolve" requires the embedding of the finite set of coalitions in the compact convex subset of "fuzzy coalitions". This survey present results obtained through this strategy.

We provide first a class of control systems governing the evolution of actions, coalitions and multilinear connectionist operators under which the architecture of a network remains viable. The controls are the "viability multipliers" of the "resource space" in which the constraints are defined. They are involved as "tensor products" of the actions of the coalitions and the viability multiplier, allowing us to encapsulate in this dynamical and multilinear framework the concept of Hebbian learning rules in neural networks in the form of "multi-Hebbian" dynamics in the evolution of connectionist operators. They are also involved in the evolution of coalitions through the "cost" of the constraints under the viability multiplier regarded as a price, describing a "nerd behavior".

We use next the viability/capturability approach for studying the problem of characterizing the dynamic core of a dynamic cooperative game defined in a characteristic function form. We define the dynamic core as a set-valued map associating with each fuzzy coalition and each time the set of imputations such that their payoffs at that time to the fuzzy coalition are larger than or equal to the one assigned by the characteristic function of the game and study it.

Keywords: Dynamic Network, Evolutionary Economics, Viability Multipliers, Fuzzy Coalitions, Connectionist Operators, Cooperative Game, Core, Shapley value

1 Introduction

Collective phenomena deal with the coordination of actions by a finite number n of players labelled $i = 1, \ldots, n$ using the architecture of a network of players, such as socio-economic networks (see for instance [7, 9, Aubin], [24, Aubin & Foray], [34, 33, Bonneuil]), neural networks (see for instance [10, 11, 8, Aubin],[20, Aubin & Burnod]) and genetic networks (see for instance [37, 36, Bonneuil], [38, Bonneuil & Saint-Pierre])). This coordinated activity requires a network of communications or connections of actions $x_i \in X_i$ ranging over n finite dimensional vector spaces X_i as well as coalitions of players.

The simplest general form of a coordination is the requirement that a relation between actions of the form $g(A(x_1, \ldots, x_n)) \in M$ must be satisfied. Here

1. $A : \prod_{i=1}^{n} X_i \mapsto Y$ is a connectionist operator relating the individual actions in a collective way,
2. $M \subset Y$ is the subset of the resource space Y and g is a map, regarded as a propagation map.

We shall study this coordination problem in a dynamic environment, by allowing actions $x(t)$ and connectionist operators $A(t)$ to evolve according to dynamical systems we shall construct later. In this case, the coordination problem takes the form

$$\forall\, t \geq 0, \;\; g(A(t)(x_1(t), \ldots, x_n(t))) \;\in\; M$$

However, in the fields of motivation under investigation, the number n of variables may be very large. Even though the connectionist operators $A(t)$ defining the "architecture" of the network are allowed to operate *a priori* on all variables $x_i(t)$, they actually operate at each instant t on a coalition $S(t) \subset N := \{1, \ldots, n\}$ of such variables, varying naturally with time according to the nature of the coordination problem.

On the other hand, a recent line of research, *dynamic cooperative game theory* has been opened by Leon Petrosjan (see for instance [59, Petrosjan] and [60, Petrosjan & Zenkevitch]), Alain Haurie ([55, Haurie]), , Georges Zeccour, Jerzy Filar and others. We quote the first lines of [47, Filar & Petrosjan] : *"Bulk of the literature dealing with cooperative games (in characteristic function form) do not address issues related to the evolution of a solution concept over time. However, most conflict situations are not "one shot" games but continue over some time horizon which may be limited* a priori *by the game rules, or terminate when* *some specified conditions are attained."* We propose here a concept of dynamic core of a dynamical fuzzy cooperative game as a set-valued map associating with each fuzzy coalition and each time the set of imputations such that their payoffs at that time to the fuzzy coalition are larger than or equal to the one assigned by the characteristic function of the game. We shall characterize this core through the (generalized) derivatives of a valuation function associated with the game, provide its explicit formula, characterize its epigraph as a viable-capture basin of the epigraph of the characteristic function of the fuzzy dynamical cooperative game, use the tangential properties of such basins for proving that the valuation function is a solution to a Hamilton-Jacobi-Isaacs partial differential equation and use this function and its derivatives for characterizing the dynamic core.

In a nutshell, this survey deals with the evolution of fuzzy coalitions for both regulate the viable architecture of a network and the evolutions of imputations in the dynamical core of a dynamical fuzzy cooperative game.

Outline

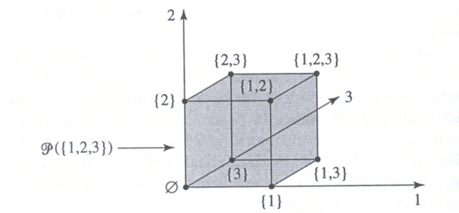
The survey is organized as follows:

1. We begin by recalling what are fuzzy coalitions in the framework of convexification procedures,
2. we proceed by studying the evolution of networks regulated by viability multipliers, showing how Hebbian rules emerge in this context
3. and by introducing fuzzy coalitions of players in this network and showing how a herd behavior emerge in this framework.
4. We next define dynamical cores of dynamical fuzzy cooperative games (with side-payments)
5. and explain briefly why the viability/capturability approach is relevant to answer the questions we have raised.
6.

2 Fuzzy Coalitions

The first definition of a coalition which comes to mind, being that of a subset of players $S \subset N$, is not adequate for tackling dynamical models of evolution of coalitions since the 2^n coalitions range over a finite set, preventing us from using analytical techniques.

One way to overcome this difficulty is to embed the family of subsets of a (discrete) set N of n players to the space $\mathbf{R}^n$:



We embed the family of subsets of a (discrete) set N of n players to the space R^n through the map χ associating with any coalition $S \in \mathcal{P}(N)$ its characteristic function $\chi_S \in \{0,1\}^n \subset \mathsf{R}^n$, since R^n can be regarded as the set of functions from N to R.
By definition, the family of fuzzy sets is the convex hull $[0,1]^n$ of the power set $\{0,1\}^n$ in R^n.

This canonical embedding is more adapted to the nature of the power set $\mathcal{P}(N)$ than to the universal embedding of a discrete set M of m elements to $\mathbf{R}^m$ by the Dirac measure associating with any $j \in M$ the jth element of the canonical basis of $\mathbf{R}^m$. The convex hull of the image of M by this embedding is the probability simplex of $\mathbf{R}^m$. Hence fuzzy sets offer a "dedicated convexification" procedure of the discrete power set $M := \mathcal{P}(N)$ instead of the universal convexification procedure of frequencies, probabilities, mixed strategies derived from its embedding in $\mathbf{R}^m = \mathbf{R}^{2^n}$.

By definition, the family of fuzzy sets[1] is the convex hull $[0,1]^n$ of the power set $\{0,1\}^n$ in $\mathbf{R}^n$. Therefore, we can write any fuzzy set in the form

$$\chi \;=\; \sum_{S \in \mathcal{P}(N)} m_S \chi_S \;\text{ where }\; m_S \geq 0 \;\;\&\;\; \sum_{S \in \mathcal{P}(N)} m_S = 1$$

The memberships are then equal to

$$\forall\, i \in N,\;\; \chi_i \;=\; \sum_{S \ni i} m_S$$

Consequently, if m_S is regarded as the probability for the set S to be formed, the membership of player i to the fuzzy set χ is the sum of the probabilities of the coalitions to which player i belongs. Player i participates fully in χ if $\chi_i = 1$, does not participate at all if $\chi_i = 0$ and participates in a fuzzy way if $\chi_i \in]0,1[$. We associate with a fuzzy coalition χ the set $P(\chi) := \{i \in N \mid \chi_i \neq 0\} \subset N$ of players i participating in the fuzzy coalition χ.

We also introduce the membership

$$\gamma_S(\chi) \;:=\; \prod_{j \in S} \chi_j$$

---

[1] This concept of fuzzy set was introduced in 1965 by L. A. Zadeh. Since then, it has been wildly successful, even in many areas outside mathematics!. We found in *"La lutte finale"*, Michel Lafon (1994), p.69 by A. Bercoff the following quotation of the late François Mitterand, president of the French Republic (1981-1995): *"Aujourd'hui, nous nageons dans la poésie pure des sous ensembles flous" ...* (Today, we swim in the pure poetry of fuzzy subsets)!

of a coalition S in the fuzzy coalition χ as the product of the memberships of players i in the coalition S. It vanishes whenever the membership of one player does and reduces to individual memberships for one player coalitions. When two coalitions are disjoint ($S \cap T = \emptyset$), then $\gamma_{S \cup T}(\chi) = \gamma_S(\chi)\gamma_T(\chi)$. In particular, for any player $i \in S$, $\gamma_S(\chi) = \chi_i \gamma_{S \setminus i}(\chi)$.

Actually, this idea of using fuzzy coalitions has already been used in the framework of static cooperative games with and without side-payments in [16, 17, Aubin], [19, Aubin, Chapter 12] and [6, Aubin, Chapter 13]. Further developments can be found in [57, Mares] and [58, Mishizaki & Sokawa], [30, 31, 32, Basile], [29, Basile, De Simone & Graziano], [48, Florenzano]). Fuzzy coalitions have also been used in dynamical models of cooperative games in [22, Aubin & Cellina, Chapter 4] and of economic theory in [9, Aubin, Chapter 5].

This idea of fuzzy sets can be adapted to more general situations relevant in game theory. We can, for instance, introduce negative memberships when players enter a coalition with aggressive intents. This is mandatory if one wants to be realistic ! A positive membership is interpreted as a cooperative participation of the player i in the coalition, while a negative membership is interpreted as a non-cooperative participation of the ith player in the generalized coalition. In what follows, one can replace the cube $[0, 1]^n$ by any product $\prod_{i=1}^{n}[\lambda_i, \mu_i]$ for describing the cooperative or noncooperative behavior of the consumers.

We can still enrich the description of the players by representing each player i by what psychologists call her 'behavior profile' as in [27, Aubin, Louis-Guerin & Zavalloni]. We consider q 'behavioral qualities' $k = 1, \ldots, q$, each with a unit of measurement. We also suppose that a behavioral quantity can be measured (evaluated) in terms of a real number (positive or negative) of units. A behavior profile is a vector $a = (a_1, \ldots, a_q) \in \mathbf{R}^q$ which specifies the quantities a_k of the q qualities k attributed to the player. Thus, instead of representing each player by a letter of the alphabet, she is described as an element of the vector space $\mathbf{R}^q$. We then suppose that each player may implement all, none, or only some of her behavioral qualities when she participates in a social coalition. Consider n players represented by their behavior profiles in $\mathbf{R}^q$. Any matrix $\chi = (\chi_i^k)$ describing the levels of participation $\chi_i^k \in [-1, +1]$ of the behavioral qualities k for the n players i is called a **social coalition**. Extension of the following results to social coalitions is straightforward.

Technically, the choice of the scaling $[0, 1]$ inherited from the tradition built on integration and measure theory is not adequate for describing convex sets. When dealing with convex sets, we have to replace the characteristic functions by indicators taking their values in $[0, +\infty]$ and take their convex combinations to provide an alternative allowing us to speak of "fuzzy" convex sets. Therefore, "toll-sets" are nonnegative cost functions assigning to each element its cost of belonging, $+\infty$ if it does not belong to the toll set. The set of elements with finite positive cost do form the "fuzzy boundary" of the toll set, the set of elements with zero cost its "core". This has been done to adapt viability theory to "fuzzy viability theory".

Actually, the Cramer transform

$$C_\mu(p) \ := \ \sup_{\chi \in \mathbf{R}^n} \left(\langle p, \chi \rangle - \log \left(\int_{\mathbf{R}^n} e^{\langle x,y \rangle} d\mu(y) \right) \right)$$

maps probability measures to toll sets. In particular, it transforms convolution products of density functions to inf-convolutions of extended functions, Gaussian functions to squares of norms, etc. See Chapter 10 of [13, Aubin] and [23, Aubin & Dordan] for more details and information on this topic.

The components of the state variable $\chi := (\chi_1, \ldots, \chi_n) \in [0,1]^n$ are the rates of participation in the fuzzy coalition χ of player $i = 1, \ldots, n$.

Hence convexification procedures and the need of using functional analysis justifies the introduction of fuzzy sets and its extensions. In the examples presented in this survey, we use only classical fuzzy sets.

3 Regulation of the Evolution of a Network

3.1 Definition of the architecture of a network

We introduce

1. n finite dimensional vector spaces X_i describing the action spaces of the players
2. a finite dimensional vector space Y regarded as a resource space and a subset $M \subset Y$ of scarce resources[2].

Definition 3.1 *The* architecture of dynamical network *involves the evolution*

1. *of actions* $x(t) := (x_1(t), \ldots, x_n(t)) \in \prod_{i=1}^n X_i$,
2. *of connectionist operators* $A_{S(t)}(t) : \prod_{i=1}^n X_i \mapsto Y$,
3. *acting on* coalitions $S(t) \subset N := \{1, \ldots, n\}$ *of the* n *players*

and requires that

$$\forall\, t \geq 0, \ \ g\left(\{A_S(t)(x(t))\}_{S \subset N}\right) \ \in \ M$$

where $g : \prod_{S \subset N} Y_S \mapsto Y$.

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[2] For simplicity, the set M of scarce resources is assumed to be constant. But sets $M(t)$ of scarce resources could also evolve through *mutational equations* and the following results can be adapted to this case. Curiously, the overall architecture is not changed when the set of available resources evolves under a mutational equation. See [5, Aubin] for more details on mutational equations.

We associate with any coalition $S \subset N$ the product $X^S := \prod_{i \in S} X_i$ and denote by $A_S \in \mathcal{L}_S(X^S, Y)$ the space of S-linear operators $A_S : X^S \mapsto Y$, i.e., operators that are linear with respect to each variable x_i, $(i \in S)$ when the other ones are fixed. Linear operators $A_i \in \mathcal{L}(X_i, Y)$ are obtained when the coalition $S := \{i\}$ is reduced to a singleton, and we identify $\mathcal{L}_\emptyset(X^\emptyset, Y) := Y$ with the vector space Y.

In order to tackle mathematically this problem, we shall

1. restrict the connectionist operators $A := \sum_{S \subset N} A_S$ to be multiaffine, i.e., the sum over all coalitions of S-linear operators[3] $A_S \in \mathcal{L}_S(X^S, Y)$,
2. allow coalitions S to become fuzzy coalitions so that they can evolve continuously.

So, a network is not only any kind of a relationship between variables, but involves both connectionist operators operating on coalitions of players.

3.2 Constructing the Dynamics

The question we raise is the following: Assume that we know the intrinsic laws of evolution of the variables x_i (independently of the constraints), of the connectionist operator $A_S(t)$ and of the coalitions $S(t)$. Is the above architecture viable under these dynamics, in the sense that the collective constraints defining the architecture of the dynamical network are satisfied at each instant.

There is no reason why let on his own, collective constraints defining the above architecture are viable under these dynamics. Then the question arises how to reestablish the viability of the system.

One may

1. either delineate those states (actions, connectionist operators, coalitions) from which begin viable evolutions,
2. or correct the dynamics of the system in order that the architecture of the dynamical network is viable under the altered dynamical system.

The first approach leads to take the viability kernel of the constrained subset of K of states (x_i, A_S, S) satisfying the constraints defining the architecture of the network. We refer to [7, 9, Aubin] for this approach. We present in this section a class of methods for correcting the dynamics without touching on the architecture of the network.

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[3] Also called (or regarded as) tensors.They are nothing other than matrices when the operators are linear instead of multilinear. Tensors are the matrices of multilinear operators, so to speak, and their “entries” depend upon several indexes instead of the two involved in matrices.

One may indeed be able, with a lot of ingeniousness and intimate knowledge of a given problem, and for "simple constraints", to derive dynamics under which the constraints are viable.

However, we can investigate whether there is a kind of mathematical factory providing classes of dynamics "correcting" the initial (intrinsic) ones in such a way that the viability of the constraints is guaranteed. One way to achieve this aim is to use the concept of "viability multipliers" $q(t)$ ranging over the dual Y^* of the resource space Y that can be used as "controls" involved for modifying the initial dynamics. This allows us to provide an **explanation of the formation and the evolution of the architecture of the network and of the active coalitions as well as the evolution of the actions themselves**.

A few words about viability multipliers are in order here: If a constrained set K is of the form

$$K := \{x \in X \quad \text{such that} \ \ h(x) \in M\}$$

where $h : X \mapsto Z := \mathbf{R}^m$ is the constrained map form the state space X to the resource space Z and $M \subset Z$ is a subset of available resources, we regard elements $u \in Z^\star = Z$ in the dual of the resource space Z (identified with Z) as **viability multipliers**, since they play a role analogous to Lagrange multipliers in optimization under constraints.

Recall that the minimization of a function $x \mapsto J(x)$ over a constrained set K is equivalent to the minimization without constraints of the function

$$x \mapsto J(x) + \sum_{k=1}^{m} \frac{\partial h_k(x)}{\partial x_j} u_k$$

for an adequate Lagrange multiplier $u \in Z^\star = Z$ in the dual of the resource space Z (identified with Z). See for instance [6, Aubin], [63, Rockafellar & Wets] among many other references on this topic.

In an analogous way, but with unrelated methods, it has been proved that a closed convex subset K is viable under the control system

$$x_j'(t) = f_j(x(t)) + \sum_{k=1}^{m} \frac{\partial h_k(x(t))}{\partial x_j} u_k(t)$$

obtained by adding to the initial dynamics a term involving regulons that belong to the dual of the same resource space Z. See for instance [22, Aubin & Cellina] and [13, 9, Aubin] below for more details. Therefore, **these viability multipliers used as regulons benefit from the same economic interpretation of virtual prices as the ones provided for Lagrange multipliers in optimization theory**.

The viability multipliers $q(t) \in Y^*$ can thus be regarded as regulons, i.e., regulation controls or parameters, or virtual prices in the language of economists. These are chosen at each instant in order that the viability constraints describing the network can be satisfied at each instant. The main theorem guarantees this possibility. Another theorem tells us how to choose at each instant such regulons (the regulation law). Even though viability multipliers do not provide all the dynamics under which a constrained set is viable, they do provide important and noticeable classes of dynamics exhibiting interesting structures that deserve to be investigated and tested in concrete situations.

3.3 An Economic Interpretation

Although the theory applies to general networks, the problem we face has an economic interpretation that may help the reader in interpreting the main results that we summarize below.

Actors here are economic agents (producers) $i = 1, \ldots, n$ ranging over the set $N := \{1, \ldots, n\}$. Each coalition $S \subset N$ of economic agents is regarded as a production unit (a firm) using resources of their agents to produce (or not produce) commodities. Each agent $i \in N$ provides a resource vector (capital, competencies, etc.) $x_i \in X$ in a resource space $X_i := \mathbb{R}^{m_i}$ used in production processes involving coalitions $S \subset N$ of economic agents (regarded as firms employing economic agents)

We describe the production process of a firm $S \subset N$ by a S-linear operator $A_S : \prod_{i=1}^{n} X_i \mapsto Y$ associating with the resources $x := (x_1, \ldots, x_n)$ provided by the economic agents a commodity $A_S(x)$. The supply constraints are described by a subset $M \subset Y$ of the commodity space, representing the set of commodities that must be produced by the firms: Condition

$$\sum_{S \subset N} A_S(t)(x(t)) \in M$$

express that at each instant, the total production must belong to M.

The connectionist operators among economic agents are the input-output production processes operating on the resources provided by the economic agents to the production units described by coalitions of economic agents. The architecture of the network is then described by the supply constraints requiring that at each instant, agents supply adequate resources to the firms in order that the production objectives are fulfilled.

When fuzzy coalitions χ_i of economic agents[4] are involved, the supply constraints are described by

$$\sum_{S \subset N} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \in M \tag{1}$$

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[4] Whenever the resources involved in production processes are proportional to the intensity of labor, one could interpret in such specific economic models the rate of participation χ_i of economic agent i as (the rate of) labor he uses in the production activity.

since the production operators are assumed to be multilinear.

3.4 Linear Connectionist Operators

We summarize the case in which there is only one player and the operator $A : X \mapsto Y$ is affine studied in [8, 9, 7, Aubin]:

$$\forall\, x \in X,\;\; A(x) \;:=\; Wx + y \;\text{ where }\; W \in \mathcal{L}(X,Y)\;\&\; y \in Y$$

The coordination problem takes the form:

$$\forall\, t \geq 0,\;\; W(t)x(t) + y(t) \;\in\; M$$

where both the state x, the resource y and the connectionist operator W evolve. These constraints are not necessarily viable under an arbitrary dynamic system of the form

$$\left\{\begin{array}{lrcl} (i) & x'(t) & = & c(x(t)) \\ (ii) & y'(t) & = & d(y(t)) \\ (iii) & W'(t) & = & \alpha(W(t)) \end{array}\right. \tag{2}$$

We can reestablish viability by involving multipliers $q \in Y^*$ ranging over the dual $Y^* := Y$ of the resource space Y to correct the initial dynamics. We denote by $W^* \in \mathcal{L}(Y^*, X^*)$ the transpose of W:

$$\forall\, q \in Y^*,\; \forall\, x \in X,\;\; \langle W^*q, x\rangle \;:=\; \langle q, Wx\rangle$$

by $x \otimes q \in \mathcal{L}(X^*, Y^*)$ the tensor product defined by

$$x \otimes q \;:\; p \in X^* := X \mapsto (x \otimes q)(p) \;:=\; \langle p, x\rangle q$$

the matrix of which is made of entries $(x \otimes q)_i^j = x_i q^j$.

The contingent cone $T_M(x)$ to $M \subset Y$ at $y \in M$ is the set of directions $v \in Y$ such that there exist sequences $h_n > 0$ converging to 0, and v_n converging to v satisfying $y + h_n v_n \in M$ for every n. The (regular) normal cone to $M \subset Y$ at $y \in M$ is defined by

$$N_M(y) := \{q \in Y^* \mid \forall\, v \in T_M(y),\; \langle q, v\rangle \leq 0\}$$

(see [25, Aubin & Frankowska] and [63, Rockafellar & Wets] for more details on these topics).

We proved that the viability of the constraints can be reestablished when the initial system (2) is replaced by the control system

$$\left\{\begin{array}{lrcl} (i) & x'(t) & = & c(x(t)) - W^*(t)q(t) \\ (ii) & y'(t) & = & d(y(t)) - q(t) \\ (iii) & W'(t) & = & \alpha(W(t)) - x(t) \otimes q(t) \\ & & & \text{where } q(t) \;\in\; N_M(W(t)x(t) + y(t)) \end{array}\right.$$

where $N_M(y) \subset Y^*$ denotes the normal cone to M at $y \in M \subset Y$ and $x \otimes q \in \mathcal{L}(X, Y^*)$ denotes the tensor product defined by

$$x \otimes q \ : \ p \in X^* := X \mapsto (x \otimes q)(x) \ := \ \langle p, x \rangle q$$

the matrix of which is made of entries $(x \otimes q)_i^j = x_i q^j$. In other words, the correction of a dynamical system for reestablishing the viability of constraints of the form $W(t)x(t) + y(t) \in M$ involves the rule proposed by Hebb in his classic book *The organization of behavior* in 1949 as the basic learning process of synaptic weight and called the Hebbian rule: Taking $\alpha(W) = 0$, the evolution of the synaptic matrix $W := (w_i^j)$ obeys the differential equation

$$\frac{d}{dt} w_i^j(t) \ = \ -x_i(t) q^j(t)$$

The Hebbian rule states that the velocity of the synaptic weight is the product of pre-synaptic activity and post-synaptic activity. Such a learning rule "pops up" (or, more pedantically, emerges) whenever the synaptic matrices are involved in regulating the system in order to maintain the "homeostatic" constraint $W(t)x(t) + y(t) \in M$. (See [10, Aubin] for more details on the relations between Hebbian rules and tensor products in the framework of neural networks).

Viability multipliers $q(t) \in Y^\star$ regulating viable evolutions satisfy the regulation law

$$\forall \, t \geq 0, \ \ q(t) \ \in \ R_M(A(t), x(t))$$

where the regulation map R_M is defined by

$$R_M(A, x) \ = \ (AA^\star + \|x\|^2 \mathsf{I})^{-1} (Ac(x) + \alpha(A)(x) - T_M(A(x)))$$

One can even require that on top of it, the viability multiplier satisfies

$$q(t) \ \in \ N_M(A(t)x(t)) \cap R_M(A(t), x(t)))$$

The norm $\|q(t)\|$ of the viability multiplier $q(t)$ measures the intensity of the viability discrepancy of the dynamic since

$$\begin{cases} (i) & \|c(x(t)) - x'(t)\| \ \leq \ \|A^*(t)\| \, \|q(t)\| \\ (ii) & \|\alpha(A(t)) - A'(t)\| \ = \ \|x(t)\| \, \|q(t)\| \end{cases}$$

When $\alpha(A) \equiv 0$, the viability multipliers with minimal norm in the regulation map provide both the smallest error $\|c(x(t)) - x'(t)\|$ and the smallest velocities of the connection matrix because $\|A'(t)\| \ = \ \|x(t)\| \|q(t)\|$. The inertia of the connection matrix, which can be regarded as an index of dynamic connectionist complexity, is proportional to the norm of the viability multiplier.

3.5 Hierarchical Architecture and Complexity

The constraints are of the form

$$A_{\mathbb{H}}^{\mathbb{H}-1}\cdots A_h^{h-1}\dots A_2^1 x_1 \;\in\; M_{\mathbb{H}}$$

This describes for instance a production process associating with the resource x_1 the intermediate outputs $x_2 := A_2^1 x_1$, which itself produces an output $x_3 := A_2^1 x_2$, and so on, until the final output $x_{\mathbb{H}} := A_{\mathbb{H}}^{\mathbb{H}-1}\cdots A_h^{h-1}\dots A_2^1 x_1$ which must belong to the production set $M_{\mathbb{H}}$.

The evolution without constraints of the commodities and the operators is governed by dynamical systems of the form

$$\left\{\begin{array}{lrcl} (i) & x_h'(t) & = & c_h(x_h(t)) \\ & & & \\ (ii) & \dfrac{d}{dt}A_{h+1}^h(t) & = & \alpha_{h+1}^h(A_h(t)) \end{array}\right.$$

The constraints

$$\forall\, t\geq 0,\;\; A_{\mathbb{H}}^{\mathbb{H}-1}(t)\cdots A_h^{h-1}(t)\dots A_2^1(t)x_1(t) \;\in\; M_{\mathbb{H}}$$

are viable under the system

$$\left\{\begin{array}{rcll} x_1'(t) & = & c_1(x_1(t)) + A_2^1(t)^{\star}(t)p^1(t) & (h=1) \\ & & & \\ x_h'(t) & = & c_h(x_h(t)) - p^{h-1}(t) + A_{h+1}^h(t)^{\star}p^h(t) & (h=1,\dots,\mathbb{H}-1) \\ & & & \\ x_{\mathbb{H}}'(t) & = & c_{\mathbb{H}}(x_{\mathbb{H}}(t)) - p^{\mathbb{H}-1}(t) & (h=\mathbb{H}) \\ & & & \\ \dfrac{d}{dt}A_{h+1}^h(t) & = & \alpha_{h+1}^h(A_h(t)) + x_h(t)\otimes p^h(t) & (h=1,\dots,\mathbb{H}-1) \end{array}\right.$$

involving viability multipliers $p^h(t)$ (intermediate "shadow price"). The input-output matrices $A_{h+1}^h(t)$ obey dynamics involving the tensor product of $x_h(t)$ and $p^h(t)$.

The viability multiplier $p^h(t)$ at level h ($h=1,\dots,\mathbb{H}-1$) both regulate the evolution at level h and send a message at upper level $h+1$.

We can tackle actually more complex hierarchical situations with non ordered hierarchies. Assume that $X := \prod_{h=1}^{\mathbb{H}}$, $Y := \prod_{k=1}^{\mathbb{K}}$ and that $A := (A_h^k)$ where $A_h^k \in \mathcal{L}(X_k, Y_h)$. We introduce a set-valued map $J : \{1,\dots,\mathbb{H}\} \rightsquigarrow \{1,\dots,\mathbb{K}\}$.

The constraints are defined by

$$\forall\, h = 1,\ldots,\mathbb{H},\quad \sum_{k\in J(h)} A_h^k(t)x_k(t) \;\in\; M_h \;\subset\; Y_h$$

We consider a system of differential equations

$$\left\{\begin{array}{lrcl} (i) & x_h'(t) & = & c_h(x_h(t)),\;\; h=1,\ldots,\mathbb{H} \\ \\ (ii) & \dfrac{d}{dt}A_h^k(t) & = & \alpha_h^k(A_h^k(t)) \end{array}\right.$$

Then the constraints

$$\forall\, h = 1,\ldots,\mathbb{H},\ldots \sum_{k\in J(h)} A_h^k(t)x_k(t) \;\in\; M_h \;\subset\; Y_h$$

are viable under the corrected system

$$\left\{\begin{array}{lrcl} (i) & x_h'(t) & = & c_h(x_h(t)) - \sum_{k\in J^{-1}(h)} A_k^h(t)^\star p^k,\;\; h=1,\ldots,\mathbb{H},\; k=1,\ldots,\mathbb{K} \\ \\ (ii) & \dfrac{d}{dt}A_h^k(t) & = & \alpha_h^k(A_h^k(t)) - x_k(t)\otimes p^h(t),\; (h,k)\in \mathrm{Graph}(J) \end{array}\right.$$

3.6 Connectionist Tensors

In order to handle more explicit and tractable formulas and results, we shall assume that the connectionist operator $A : X := \prod_{i=1}^n X_i \rightsquigarrow Y$ is multiaffine.

For defining such a multiaffine operator, we associate with any coalition $S \subset N$ its characteristic function $\chi_S : N \mapsto \mathbf{R}$ associating with any $i \in N$

$$\chi_S(i) \;:=\; \left\{\begin{array}{ll} 1 & \text{if} \;\; i\in S \\ 0 & \text{if} \;\; i\notin S \end{array}\right.$$

It defines a linear operator $\chi_S\circ \in \mathcal{L}\left(\prod_{i=1}^n X_i, \prod_{i=1}^n X_i\right)$ that associates with any $x = (x_1,\ldots,x_n) \in \prod_{i=1}^n X_i$ the sequence $\chi_S \circ x \in \mathbf{R}^n$ defined by

$$\forall\, i = 1,\ldots,n,\;\; (\chi_S\circ x)_i \;:=\; \left\{\begin{array}{ll} x_i & \text{if} \;\; i\in S \\ 0 & \text{if} \;\; i\notin S \end{array}\right.$$

We associate with any coalition $S \subset N$ the subspace

$$X^S \;:=\; x_S \circ \prod_{i=1}^{n} X_i \;=\; \left\{ x \in \prod_{i=1}^{n} X_i \quad \text{such that } \; \forall\, i \notin S, \;\; x_i = 0 \right\}$$

since $x_S \circ$ is nothing other that the canonical projector from $\prod_{i=1}^{n} X_i$ onto X^S. In particular, $X^N := \prod_{i=1}^{n} X_i$ and $X^{\emptyset} := \{0\}$.

Let Y be another finite dimensional vector space. We associate with any coalition $S \subset N$ the space $\mathcal{L}_S(X^S, Y)$ of S-linear operators A_S. We extend such a S-linear operator A_S to a n-linear operator (again denoted by) $A_S \in \mathcal{L}_n\left(\prod_{i=1}^{n} X_i, Y\right)$ defined by:

$$\forall\, x \in \prod_{i=1}^{n} X_i, \;\; A_S(x) \;=\; A_S(x_1, \ldots, x_n) \;:=\; A_S(\chi_S \circ x)$$

A multiaffine operator $A \in \mathcal{A}_n\left(\prod_{i=1}^{n} X_i, Y\right)$ is a sum of S-linear operators $A_S \in \mathcal{L}_S(X^S, Y)$ when S ranges over the family of coalitions:

$$A(x_1, \ldots, x_n) \;:=\; \sum_{S \subset N} A_S(\chi_S \circ x) \;=\; \sum_{S \subset N} A_S(x)$$

We identify $A_{\emptyset}$ with a constant $A_{\emptyset} \in Y$.

Hence the collective constraint linking multiaffine operators and actions can be written in the form

$$\forall\, t \geq 0, \;\; \sum_{S \subset N} A_S(t)(x(t)) \;\in\; M$$

For any $i \in S$, we shall denote by $(x_{-i}, u_i) \in X^N$ the sequence $y \in X^N$ where $y_j := x_j$ when $j \neq i$ and $y_i = u_i$ when $j = i$. The linear operator $A_S\left(x_{-i}\right) \in \mathcal{L}(X_i, Y)$ is defined by $u_i \mapsto A_S(x_{-i})u_i := A_S(x_{-i}, u_i)$. We shall use its transpose $A_S\left(x_{-i}\right)^{\star} \in \mathcal{L}(Y^{\star}, X_i^{\star})$ defined by

$$\forall\, q \in Y^{\star}, \; \forall\, u_i \in X_i, \;\; \left\langle A_S\left(x_{-i}\right)^{\star} q, u_i \right\rangle \;=\; \left\langle q, A_S\left(x_{-i}\right) u_i \right\rangle$$

We associate with $q \in Y^{\star}$ and elements $x_i \in X_i$ the multilinear operator[5]

$$x_1 \otimes \cdots \otimes x_n \otimes q \;\in\; \mathcal{L}_n\left(\prod_{i=1}^{n} X_i^{\star}, Y^{\star}\right)$$

---

[5] We recall that the space $\mathcal{L}_n\left(\prod_{i=1}^{n} X_i, Y\right)$ of n-linear operators from $\prod_{i=1}^{n} X_i$ to Y is isometric to the tensor product $\bigotimes_{i=1}^{n} X_i^{\star} \otimes Y$, the dual of which is $\bigotimes_{i=1}^{n} X_i \otimes Y^{\star}$, that is isometric with $\mathcal{L}_n\left(\prod_{i=1}^{n} X_i^{\star}, Y^{\star}\right)$.

associating with any $p := (p_1, \ldots, p_n) \in \prod_{i=1}^n X_i^*$ the element $\left(\prod_{i=1}^{n} \langle p_i, x_i \rangle\right) q$:

$$x_1 \otimes \cdots \otimes x_n \otimes q : p := (p_1, \ldots, p_n) \in \prod_{i=1}^{n} X_i^* \mapsto (x_1 \otimes \cdots \otimes x_n \otimes q)(p) := \left(\prod_{i=1}^{n} \langle p_i, x_i \rangle\right) q \in Y^*$$

This multilinear operator $x_1 \otimes \cdots \otimes x_n \otimes q$ is called the tensor product of the x_i's and q.

We recall that the duality product on $\mathcal{L}_n\left(\prod_{i=1}^n X_i^*, Y^*\right) \times \mathcal{L}_n\left(\prod_{i=1}^n X_i, Y\right)$ for pairs $(x_1 \otimes \cdots \otimes x_n \otimes q, A)$ can be written in the form:

$$\langle x_1 \otimes \cdots \otimes x_n \otimes q, A \rangle \ := \ \langle q, A(x_1, \ldots, x_n) \rangle$$

3.7 Multi-Hebbian Learning Process

Assume that we start with intrinsic dynamics of the actions x_i, the resources y, the connectionist matrices W and the fuzzy coalitions χ:

$$\begin{cases} (i) & x_i'(t) \ = \ c_i(x(t)), \ i = 1, \ldots, n \\ (ii) & A_S'(t) \ = \ \alpha_S(A(t)), \ S \subset N \end{cases}$$

Using viability multipliers, we can modify the above dynamics by introducing regulons that are elements $q \in Y^*$ of the dual Y^* of the space Y:

Theorem 3.2 *Assume that the functions c_i, κ_i and α_S are continuous and that $M \subset Y$ are closed. Then the constraints*

$$\forall \, t \geq 0, \ \sum_{S \subset N} A_S(t)(x(t)) \ \in \ M$$

are viable under the control system

$$\begin{cases} (i) & x_i'(t) \ = \ c_i(x_i(t)) - \displaystyle\sum_{S \ni i} A_S(t)(x_{-i}(t))^* q(t), \ i = 1, \ldots, n \\ & \\ (ii) & A_S'(t) \ = \ \alpha_S(A(t)) - \left(\displaystyle\bigotimes_{j \in S} x_j(t)\right) \otimes q(t), \ S \subset N \\ & \text{where } q(t) \in N_M\left(\sum_{S \subset N} A_S(t)(x(t))\right) \end{cases}$$

Remark: Multi-Hebbian Rule — When we regard the multilinear operator A_S as a tensor product of components $A^j_{S_{\Pi_{i\in S} i_k}}$, $j = 1, \ldots, p$, $i_k = 1, \ldots, n_i$, $i \in S$, differential equation (ii) can be written in the form: $\forall\, i \in S$, $j = 1, \ldots, p$, $k = 1, \ldots, n_i$,

$$\frac{d}{dt} A^j_{S_{\Pi_{i\in S} i_k}} \;=\; \alpha_{S_{\Pi_{i\in S} i_k}}(A(t)) - \left(\prod_{i\in S} x_{i_k}(t)\right) q^j(t)$$

The correction term of the component $A^j_{S_{\Pi_{i\in S} i_k}}$ of the S-linear operator is the product of the components $x_{i_k}(t)$ actions x_i in the coalition S and of the component q^j of the viability multiplier. *This can be regarded as a multi-Hebbian rule in neural network learning algorithms*, since for linear operators, we find the product of the component x_k of the pre-synaptic action and the component q^j of the post-synaptic action. $\square$

Indeed, when the vector spaces $X_i := \mathbf{R}^{n_i}$ are supplied with basis e^{i_k}, $k = 1, \ldots, n_i$, when we denote by $e^*_{i_k}$ their dual basis, and when $Y := \mathbf{R}^p$ is supplied with a basis f^j, and its dual supplied with the dual basis f^*_j, then the tensor products $\left(\bigotimes_{i\in S} e^{i_k}\right) \otimes f^*_j$ ($j = 1, \ldots, p$, $k = 1, \ldots, n_i$) form a basis of $\mathcal{L}_S\left(X^{S^*}, Y^*\right)$.

Hence the components of the tensor product $\left(\bigotimes_{i\in S} x_i\right) \otimes q$ in this basis are the products $\left(\prod_{i\in S} x_{i_k}\right) q^j$ of the components q^j of q and x_{i_k} of the x_i's, where $q^j := \langle q, f^j\rangle$ and $x_{i_k} := \langle e^*_{i_k}, x_i\rangle$. Indeed, we can write

$$\left(\bigotimes_{i\in S} x_i\right) \otimes q \;=\; \sum_{j=1}^{p} \sum_{i\in S} \sum_{k=1}^{n_i} \left(\langle q, f^j\rangle \prod_{i\in S} \langle e^*_{i_k}, x_i\rangle\right) \left(\bigotimes_{i=1}^{n} e^{i_k}\right) \otimes f^*_j$$

4 Regulation Involving Fuzzy Coalitions

Let $A \in \mathcal{A}_n\left(\prod_{i=1}^n X_i, Y\right)$, a sum of S-linear operators $A_S \in \mathcal{L}_S(X^S, Y)$ when S ranges over the family of coalitions, be a multiaffine operator.

When χ is a fuzzy coalition, we observe that

$$A(\chi \circ x) \;=\; \sum_{S\subset P(\chi)} \gamma_S(\chi) A_S(x) \;=\; \sum_{S\subset P(\chi)} \left(\prod_{j\in S} \chi_j\right) A_S(x)$$

We wish to encapsulate the idea that at each instant, only a number of fuzzy coalitions χ are active. Hence the collective constraint linking multiaffine operators, fuzzy coalitions and actions can be written in the form

$$\forall\, t \geq 0, \quad \sum_{S \subset P(\chi(t))} \gamma_S(\chi(t)) A_S(t)(x(t)) \; = \; \sum_{S \subset P(\chi(t))} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \; \in \; M$$

4.1 Constructing Viable Dynamics

Assume that we start with intrinsic dynamics of the actions x_i, the resources y, the connectionist matrices W and the fuzzy coalitions χ:

$$\left\{ \begin{array}{lrcl} (i) & x_i'(t) & = & c_i(x(t)),\; i = 1, \ldots, n \\ (ii) & \chi_i'(t) & = & \kappa_i(\chi(t)),\; i = 1, \ldots, n \\ (iii) & A_S'(t) & = & \alpha_S(A(t)),\; S \subset N \end{array} \right.$$

Using viability multipliers, we can modify the above dynamics by introducing regulons that are elements $q \in Y^*$ of the dual Y^* of the space Y:

Theorem 4.1 *Assume that the functions c_i, κ_i and α_S are continuous and that $M \subset Y$ are closed. Then the constraints*

$$\forall\, t \geq 0, \quad \sum_{S \subset P(\chi(t))} A_S(t)(\chi(t) \circ x(t)) \; = \; \sum_{S \subset P(\chi(t))} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \; \in \; M$$

are viable under the control system

$$\left\{ \begin{array}{ll} (i) & x_i'(t) \; = \; c_i(x_i(t)) - \displaystyle\sum_{S \ni i} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x_{-i}(t))^* q(t),\; i = 1, \ldots, n \\ & \\ (ii) & \chi_i'(t) \; = \; \kappa_i(\chi(t)) - \displaystyle\sum_{S \ni i} \left(\prod_{j \in S \setminus i} \chi_j(t) \right) \langle q(t), A_S(t)\,(x(t)) \rangle,\; i = 1, \ldots, n \\ & \\ (iii) & A_S'(t) \; = \; \alpha_S(A(t)) - \left(\displaystyle\prod_{j \in S} \chi_j(t) \right) \left(\bigotimes_{j \in S} x_j(t) \right) \otimes q(t),\; S \subset N \\ & \quad \text{where } q(t) \in N_M(\sum_{S \subset P(\chi(t))} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)))) \end{array} \right.$$

Let us comment on these formulas. First, the viability multipliers $q(t) \in Y^*$ can be regarded as regulons, i.e., regulation controls or parameters, or virtual prices in the language of economists. They are chosen adequately at each instant in order that the viability constraints describing the network can be satisfied at each instant, and the above theorem guarantees this possibility. The next section tells us how to choose at each instant such regulons (the regulation law).

For each player i, the velocities $x_i'(t)$ of the state and the velocities $\chi_i'(t)$ of its membership in the fuzzy coalition $\chi(t)$ are corrected by subtracting

1. the sum over all coalitions S to which he belongs of the $A_S(t)(x_{-i}(t))^* q(t)$ weighted by the membership $\gamma_S(\chi(t))$:

   $$x_i'(t) \; = \; c_i(x_i(t)) - \sum_{S \ni i} \gamma_S(\chi(t)) A_S(t)(x_{-i}(t))^* q(t)$$

2. the sum over all coalitions S to which he belongs of the costs $\langle q(t), A_S(t)\,(x(t))\rangle$ of the constraints associated with connectionist tensor A_S of the coalition S weighted by the membership $\gamma_{S\backslash i}(\chi(t))$:

   $$\chi_i'(t) \; = \; \kappa_i(\chi(t)) - \sum_{S \ni i} \gamma_{S\backslash i}(\chi(t)) \, \langle q(t), A_S(t)\,(x(t))\rangle$$

   The (algebraic) increase of player i's membership in the fuzzy coalition aggregates over all coalitions to which he belongs the cost of their constraints weighted by the products of memberships of the other players in the coalition.

   It can be interpreted as an incentive for economic agents to increase or decrease his participation in the economy in terms of the cost of constraints and of the membership of other economic agents, encapsulating a mimetic — or "herd", panurgean — behavior (from a famous story by François Rabelais (1483-1553), where Panurge sent overboard the head sheep, followed by the whole herd).

	Panurge ... jette en pleine mer son mouton criant et bellant. Tous les aultres moutons, crians et bellans en pareille intonation, commencerent soy jecter et saulter en mer après, à la file ... comme vous sçavez estre du mouton le naturel, tous jours suyvre le premier, quelque part qu'il aille. Aussi li dict Aristoteles, lib. 9, de Histo. animal. estre le plus sot et inepte animant du monde.

As for the correction of the velocities of the connectionist tensors A_S, their correction is a weighted "multi-Hebbian" rule: for each component $A^j_{S_{\Pi_{i\in S} i_k}}$ of A_S, the correction term is the product of the membership $\gamma S(\chi(t))$ of the coalition S, of the components $x_{i_k}(t)$ and of the component $q^j(t)$ of the regulon:

$$\frac{d}{dt}A^j_{S_{\Pi_{i\in S}i_k}} = \alpha_{S_{\Pi_{i\in S}i_k}}(A(t)) - \gamma_S(\chi(t))\left(\prod_{i\in S} x_{i_k}(t)\right) q^j(t)$$

4.2 The Regulation Map

Actually, the viability multipliers $q(t)$ regulating viable evolutions of the actions $x_i(t)$, the fuzzy coalitions $\chi(t)$ and the multiaffine operators $A(t)$ obey the regulation law (an "adjustment law", in the vocabulary of economists) of the form

$$\forall\, t \geq 0, \;\; q(t) \;\in\; R_M(x(t), \chi(t), A(t))$$

where $R_M : X^N \times \mathbf{R}^n \times \mathcal{A}_n(X^N, Y) \rightsquigarrow Y^\star$ is the regulation map R_M that we shall compute.

For this purpose, we introduce the operator $h : X^N \times \mathbf{R}^n \times \mathcal{A}_n(X^N, Y)$ defined by

$$h(x, \chi, A) \;:=\; \sum_{S\subset N} A_S(\chi \circ x)$$

and the linear operator $H(x, \chi, A) : Y^\star := Y \mapsto Y$ defined by:

$$\left\{\begin{array}{l} H(x,\chi,A) \;:=\; \displaystyle\sum_{S\subset N}\left(\prod_{j\in S}\chi_j^2\|x_j\|^2\right)\mathbf{I} \\ + \displaystyle\sum_{R,S\subset N}\sum_{i\in R\cap S}\big(\gamma_R(\chi)\gamma_S(\chi)A_R(x_{-i})A_S(x_{-i})^\star + \gamma_{R\backslash i}(\chi)\gamma_{S\backslash i}(\chi)A_R(x)\otimes A_S(x)\big)\end{array}\right.$$

Then the regulation map is defined by

$$\left\{\begin{array}{l} R_M(x,\chi,A) \;:=\; H(x,\chi,A)^{-1} \\ \left(\displaystyle\sum_{S\subset N}\left(\alpha_S(A)(x) + \sum_{i\in S}\big(\gamma_S(\chi)A_S(x_{-i}, c_i(x)) + \gamma_{S\backslash i}(\chi)\kappa_i(\chi)A_S(x)\big)\right) - T_M(h(x,\chi,A))\right)\end{array}\right.$$

Indeed, the regulation map R_M associates with any (x, χ, A) the subset $R_M(x, \chi, A)$ of $q \in Y^\star$ such that

$$h'(x,\chi,A)((c(x),\kappa(\chi),\alpha(A)) - h'(x,\chi,A)^\star q) \;\in\; \overline{\mathrm{co}}\,(T_M(h(x)))$$

We next observe that

$$h'(x,\chi,A)h'(x,\chi,A)^\star \;=\; H(x,\chi,A)$$

and that

$$\left\{\begin{array}{l} h'(x,\chi,A)(c(x),\kappa(\chi),\alpha(A)) \\ = \displaystyle\sum_{S\subset N}\left(\alpha_S(A)(x)+\sum_{i\in S}\left(\gamma_S(\chi)A_S(x_{-i},c_i(x))+\gamma_{S\setminus i}(\chi)\kappa_i(\chi)A_S(x)\right)\right) \end{array}\right.$$

Remark:Links between viability and Lagrange multipliers —

The point made in this paper is to show how the mathematical methods presented in a general way can be useful in designing other models, as the Lagrange multiplier rule does in the static framework. By comparison, we see that if we minimize a collective utility function:

$$\sum_{i=1}^{n}\mathbf{u}_i(x_i)+\sum_{i=1}^{n}\mathbf{v}_i(\chi_i)+\sum_{S\subset N}\mathbf{w}_S(A_S)$$

under constraints (1), then first-order optimality conditions at a optimum $((x_i)_i,(\chi_i)_i,(A_S)_{S\subset N})$ imply the existence of Lagrange multipliers p such that:

$$\left\{\begin{array}{lll} \nabla\mathbf{u}_i(x_i) & = & \displaystyle\sum_{S\ni i}\left(\prod_{j\in S}\chi_j\right)A_S(x_{-i}(t))^*p,\ i=1,\ldots,n \\ \nabla\mathbf{v}_i(\chi_i) & = & \displaystyle\sum_{S\ni i}\left(\prod_{j\in S\setminus i}\chi_j\right)\langle p, A_S\left(x\right)\rangle\,,\ i=1,\ldots,n \qquad \blacksquare \\ \nabla\mathbf{w}_S(A_S) & = & \displaystyle\left(\prod_{j\in S}\chi_j\right)\left(\bigotimes_{j\in S}x_j\right)\otimes p,\ S\subset N \end{array}\right.$$

5 Dynamical Fuzzy Cooperative Games under Tychastic Uncertainty

5.1 Static Fuzzy Cooperative Games

Definition 5.1 *A Fuzzy game with side-payments is defined by a characteristic function* $\mathbf{u}:[0,1]^n\mapsto\mathbf{R}_+$ *of a fuzzy game assumed to be positively homogenous.*

When the characteristic function of the static cooperative game $\mathbf{u}$ is concave, positively homogeneous and continuous on the interior of $\mathbf{R}_+^n$, one checks[6] that the generalized gradient

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[6] See [16, 17, Aubin], [19, Aubin, Chapter 12] and [6, Aubin, Chapter 13].

$\partial \mathbf{u}(\chi_N)$ is not empty and coincides with the subset of imputations $p := (p_1, \ldots, p_n) \in \mathbf{R}^n_+$ accepted by all fuzzy coalitions in the sense that

$$\forall\, \chi \in [0,1]^n, \ \ \langle p, \chi \rangle \ = \ \sum_{i=1}^{n} p_i \chi_i \ \geq \ \mathbf{u}(\chi) \tag{3}$$

and that, for the grand coalition $\chi_N := (1, \ldots, 1)$,

$$\langle p, \chi_N \rangle \ = \ \sum_{i=1}^{n} p_i \ = \ \mathbf{u}(\chi_N)$$

It has been shown that in the framework of static cooperative games with side payments involving fuzzy coalitions, *the concepts of Shapley value and core coincide with the (generalized) gradient* $\partial \mathbf{u}(\chi_N)$ *of the "characteristic function"* $\mathbf{u} : [0,1]^n \mapsto \mathbf{R}_+$ *at the "grand coalition"* $\chi_N := (1, \ldots, 1)$*, the characteristic function of* $N := \{1, 2, \ldots, n\}$. The differences between these concepts for usual games is explained by the different ways one "fuzzyfies" a characteristic function defined on the set of usual coalitions.

5.2 Three examples of Game Rules

In a dynamical context, (fuzzy) coalitions evolve, so that static conditions (3) should be replaced by conditions[7] stating that for any evolution $t \mapsto x(t)$ of fuzzy coalitions, the payoff $y(t) := \langle p(t), \chi(t) \rangle$ should be larger than or equal to $\mathbf{u}(\chi(t))$ according (at least) to one of the three following rules:

1. at a prescribed final time T of the end of the game:

$$y(T) \ := \ \sum_{i=1}^{n} p_i(T) \chi_i(T) \ \geq \ \mathbf{u}(\chi(T))$$

2. during the whole time span of the game:

$$\forall\, t \in [0,T], \ \ y(t) \ := \ \sum_{i=1}^{n} p_i(t) \chi_i(t) \ \geq \ \mathbf{u}(\chi(t))$$

3. at the first winning time $t^* \in [0,T]$ when

$$y(t^*) \ := \ \sum_{i=1}^{n} p_i(t^*) \chi_i(t^*) \ \geq \ \mathbf{u}(\chi(t^*))$$

   at which time the game stops.

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[7] Naturally, the privileged role played by the grand coalition in the static case must be abandoned, since the coalitions evolve, so that the grand coalition eventually loses its capital status.

Summarizing, the above conditions require to find — for each of the above three rules of the game — an evolution of an imputation $p(t) \in \mathbf{R}^n$ such that, for all evolutions of fuzzy coalitions $\chi(t) \in [0,1]^n$ starting at χ, the corresponding rule of the game

$$\left\{ \begin{array}{ll} i) & \sum_{i=1}^n p_i(T)\chi_i(T) \ \geq \ \mathbf{u}(\chi(T)) \\ ii) & \forall\, t \in [0,T], \ \sum_{i=1}^n p_i(t)\chi_i(t) \ \geq \ \mathbf{u}(\chi(t)) \\ iii) & \exists\, t^\star \in [0,T] \ \text{ such that } \sum_{i=1}^n p_i(t^\star)\chi_i(t^\star) \ \geq \ \mathbf{u}(\chi(t^\star)) \end{array} \right. \tag{4}$$

must be satisfied.

Therefore, for each one of the above three rules of the game (4), a concept of dynamical core should provide a set-valued map $\Gamma : \mathbf{R}_+ \times [0,1]^n \rightsquigarrow \mathbf{R}^n$ associating with each time t and any fuzzy coalition χ a set $\Gamma(t,\chi)$ of imputations $p \in \mathbf{R}^n_+$ such that, taking $p(t) \in \Gamma(T-t,\chi(t))$, and in particular, $p(0) \in \Gamma(T,\chi(0))$, the chosen above condition is satisfied. This is the purpose of this study.

5.3 A General Class of Game Rules

Actually, in order to treat the three rules of the game (4) as particular cases of a more general framework, we introduce two nonnegative extended functions $\mathbf{b}$ and $\mathbf{c}$ (characteristic functions of the cooperative games) satisfying

$$\forall\, (t,\chi) \in \mathbf{R}_+ \times \mathbf{R}^n_+ \times \mathbf{R}^n, \ \ 0 \ \leq \ \mathbf{b}(t,\chi) \ \leq \ \mathbf{c}(t,\chi) \ \leq \ +\infty$$

By associating with the initial characteristic function $\mathbf{u}$ of the game adequate pairs $(\mathbf{b},\mathbf{c})$ of extended functions, we shall replace the requirements (4) by the requirement

$$\left\{ \begin{array}{ll} i) & \forall\, t \in [0,t^\star], \ \ y(t) \geq \mathbf{b}(T-t,\chi(t)) \ \ \text{(dynamical constraints)} \\ ii) & y(t^\star) \geq \mathbf{c}(T-t^\star,\chi(t^\star)) \ \ \text{(objective)} \end{array} \right. \tag{5}$$

We extend the functions $\mathbf{b}$ and $\mathbf{c}$ as functions from $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$ to $\mathbf{R}_+ \cup \{+\infty\}$ by setting

$$\forall\, t < 0, \ \ \mathbf{b}(t,\chi) \ = \ \mathbf{c}(t,\chi) \ = \ +\infty$$

so that nonnegativity constraints on time are automatically taken into account.

For instance, *problems with prescribed final time are obtained with objective functions satisfying the condition*

$$\forall\, t > 0, \ \ \mathbf{c}(t,\chi) \ := \ +\infty$$

In this case, $t^\star = T$ and condition (5) boils down to

$$\left\{ \begin{array}{ll} i) & \forall\, t \in [0,T], \ \ y(t) \geq b(T-t,\chi(t)) \\ ii) & y(T) \geq c(0,\chi(T)) \end{array} \right.$$

Indeed, since $y(t^*)$ is finite and since $\mathbf{c}(T-t^*, \chi(t^*))$ is infinite whenever $T-t^* > 0$, we infer from inequality (5)ii) that $T-t^*$ must be equal to 0. ■

Allowing the characteristic functions to take infinite values (i.e., to be extended), allows us to acclimate many examples.

For example, *the three rules (4) associated with a same characteristic function* $\mathbf{u} : [0,1]^n \mapsto \mathbf{R} \cup \{+\infty\}$ *can be written in the form (5) by adequate choices of pairs* $(\mathbf{b}, \mathbf{c})$ *of functions associated with* $\mathbf{u}$. Indeed, denoting by u_∞ the function defined by

$$\mathbf{u}_\infty(t,\chi) := \begin{cases} \mathbf{u}(\chi) & \text{if } t = 0 \\ +\infty & \text{if } t > 0 \end{cases}$$

and by $\mathbf{0}$ the function defined by

$$\mathbf{0}(t,\chi) = \begin{cases} 0 & \text{if } \quad t \geq 0, \\ +\infty & \text{if not} \end{cases}$$

we can recover the three rules of the game

1. We take $\mathbf{b}(t,\chi) := \mathbf{0}(t,\chi)$ and $\mathbf{c}(t,\chi) = \mathbf{u}_\infty(t,\chi)$, we obtain the prescribed final time rule (4)i).
2. We take $\mathbf{b}(t,\chi) := \mathbf{u}(\chi)$ and $\mathbf{c}(t,\chi) := \mathbf{u}_\infty(t,\chi)$, we obtain the span time rule (4)ii).
3. We take $\mathbf{b}(t,\chi) := \mathbf{0}(t,\chi)$ and $\mathbf{c}(t,\chi) = \mathbf{u}(\chi)$, we obtain the first winning time rule (4)iii).

5.4 Dynamics of Fuzzy Cooperative Games

Naturally, games are played under uncertainty. In games arising social or biological sciences, uncertainty is rarely od a probabilistic and stochastic nature (with statistical regularity), but of a tychastic nature, according to a terminology borrowed to Charles Peirce.

Next, we define the dynamics of the coalitions and of the imputations, assumed to be given.

1. the evolution of coalitions $\chi(t) \in \mathbf{R}^n$ is governed by differential inclusions

   $$\chi'(t) := f(\chi(t), v(t)) \text{ where } v(t) \in Q(\chi(t))$$

   where $v(t)$ are tyches,

State-dependent uncertainty can also be translated mathematically by parameters on which actors, agents, decision makers, etc. have no controls. These parameters are often perturbations, disturbances (as in "robust control" or "differential games against nature") or more generally, **tyches** (meaning "chance" in classical Greek, from the Goddess Tyche) ranging over a state-dependent **tychastic map**. They could be called "random variables" if this vocabulary were not already confiscated by probabilists. This is why we borrow the term of *tychastic evolution* to Charles Peirce who introduced it in a paper published in 1893 under the title *evolutionary love*. One can prove that stochastic viability is a (very) particular case of tychastic viability. The size of the tychastic map captures mathematically the concept of "**versatility (tychastic volatility)**" — instead of "(stochastic) volatility": **The larger the graph of the tychastic map, the more "versatile" the system.**

2. **static constraints**

   $$\forall\, \chi \in [0,1]^n, \;\; p \;\in\; P(\chi) \;\subset\; \mathbf{R}^n_+$$

   and **dynamic constraints** on the velocities of the imputations $p(t) \in \mathbf{R}^n_+$ of the form

   $$\langle p'(t), \chi(t)\rangle \;=\; -\mathbf{m}(\chi(t), p(t), v(t))\langle p(t), \chi(t)\rangle$$

   stating that the cost $\langle p', \chi\rangle$ of the instantaneous change of imputation of a coalition is proportional to it by a discount factor $\mathbf{m}(\chi, p)$

3. from which we deduce the velocity $y'(t) = \langle p(t), f(\chi(t), v(t))\rangle - \mathbf{m}(\chi(t), p(t))y(t)$ of the payoff $y(t) := \langle p(t), \chi(t)\rangle$ of the fuzzy coalition $\chi(t)$.

The evolution of the fuzzy coalitions is thus parameterized by imputations and tyches, i.e., is governed by a dynamic game

$$\left\{\begin{array}{lll} i) & \chi'(t) \;=\; f(\chi(t), v(t)) \\ ii) & y'(t) \;=\; \langle p(t), f(\chi(t), v(t))\rangle - \mathbf{m}(\chi(t), p(t))y(t) \\ iii) & \text{where } p(t) \;\in\; P(\chi(t)) \;\;\&\;\; v(t) \;\in\; Q(\chi(t)) \end{array}\right. \tag{6}$$

A **feedback** $\widetilde{p}$ is a **selection** of the set-valued map P in the sense that for any $\chi \in [0,1]^n$, $\widetilde{p}(\chi) \in P(\chi)$. We thus associate with any feedback $\widetilde{p}$ the set $\mathcal{C}_{\widetilde{p}}(\chi)$ of triples $(\chi(\cdot), y(\cdot), v(\cdot))$ solutions to

$$\left\{\begin{array}{lll} i) & \chi'(t) \;=\; f(\chi(t), v(t)) \\ ii) & y'(t) \;=\; \langle \widetilde{p}(\chi(t)), f(\chi(t), v(t))\rangle - y(t)\mathbf{m}(\chi(t), \widetilde{p}(\chi(t)), v(t)) \\ & \text{where } \;v(t) \;\in\; Q(\chi(t)) \end{array}\right. \tag{7}$$

5.5 Valuation of the Dynamical Game

We shall characterize the dynamical core of the fuzzy dynamical cooperative game in terms of the derivatives of a valuation function that we now define.

For each rule of the game (5), the set $\mathcal{V}^\sharp$ of initial conditions (T, χ, y) such that there exists a feedback $\chi \mapsto \widetilde{p}(\chi) \in P(\chi)$ such that, for all tyches $t \in [0, T] \mapsto v(t) \in Q(\chi(t))$, for all solutions to system (7) of differential equations satisfying $\chi(0) = \chi$, $y(0) = y$, the corresponding condition (5) is satisfied, is called the guaranteed valuation set[8].

Knowing it, we deduce the valuation function

$$V^\sharp(T, \chi) \ := \ \inf\{y \mid (T, \chi, y) \in \mathcal{V}^\sharp\}$$

providing the cheapest initial payoff allowing to satisfy the viability/capturability conditions (5). It satisfies the initial condition:

$$V^\sharp(0, \chi) \ := \ \mathbf{u}(\chi)$$

In each of the three cases, we shall compute explicitly the valuation functions as infsup of underlying criteria we shall uncover. For that purpose, we associate with the characteristic function $\mathbf{u} : [0, 1]^n \mapsto \mathbf{R} \cup \{+\infty\}$ of the dynamical cooperative game the functional

$$\begin{cases} J_{\mathbf{u}}(t; (\chi(\cdot), v(\cdot)); \widetilde{p})(\chi) \ := \ e^{\int_0^t \mathbf{m}(\chi(s), \widetilde{p}(\chi(s)), v(s)) ds} \mathbf{u}(\chi(t)) \\ - \displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(\chi(s), \widetilde{p}(\chi(s)), v(s)) ds} \langle \widetilde{p}(\chi(\tau)), f(\chi(\tau), v(\tau)) \rangle d\tau \end{cases}$$

We shall associate with it and with each of the three rules of the game 4 the three corresponding valuation functions:

1. **prescribed end rule**: We obtain

$$V^\sharp_{(\mathbf{0}, \mathbf{u}_\infty)}(T, \chi) \ := \ \inf_{\widetilde{p}(\chi) \in P(\chi)} \ \sup_{(\chi(\cdot), v(\cdot)) \in \mathcal{C}_{\widetilde{p}}(\chi)} J_{\mathbf{u}}(T; (\chi(\cdot), v(\cdot)); \widetilde{p})(\chi) \qquad (8)$$

2. **time span rule**: We obtain

$$V^\sharp_{(\mathbf{u}, \mathbf{u}_\infty)}(T, \chi) \ := \ \inf_{\widetilde{p}(\chi) \in P(\chi)} \ \sup_{(\chi(\cdot), v(\cdot)) \in \mathcal{C}_{\widetilde{p}}(\chi)} \ \sup_{t \in [0, T]} J_{\mathbf{u}}(t; (\chi(\cdot), v(\cdot)); \widetilde{p})(\chi) \qquad (9)$$

---

[8] One can also define the conditional valuation set $\mathcal{V}^\flat$ of initial conditions (T, χ, y) such that for all tyches v, there exists an evolution of the imputation $p(\cdot)$ such that viability/capturability conditions (5) are satisfied. We omit this study for the sake of brevity, since it is parallel to the one of guaranteed valuation sets.

3. **first winning time rule**: We obtain

$$V^{\sharp}_{(\mathbf{0},\mathbf{u})}(T,\chi) \ := \ \inf_{\widetilde{p}(\chi)\in P(\chi)} \ \sup_{(\chi(\cdot),v(\cdot))\in \mathcal{C}_{\widetilde{p}}(\chi)} \ \inf_{t\in[0,T]} J_{\mathbf{u}}(t;(\chi(\cdot),v(\cdot));\widetilde{p})(\chi) \tag{10}$$

A general formula for game rules 5 does exist, but is too involved to be reproduced in this survey.

5.6 Hamilton-Jacobi Equations and Dynamical Core

Although these functions are only lower semicontinuous, one can define epiderivatives of lower semicontinuous functions (or generalized gradients) in adequate ways and compute the core Γ: for instance, when the valuation function is differentiable, we shall prove that Γ associates with any $(t,\chi)\in \mathbf{R}_+\times\mathbf{R}^n$ the subset $\Gamma(t,\chi)$ of imputations $p\in P(\chi)$ satisfying

$$\sup_{v\in Q(\chi)}\left(\sum_{i=1}^{n}\left(\frac{\partial V^{\sharp}(t,\chi)}{\partial\chi_i}-p_i\right)f_i(\chi,v)+\mathbf{m}(\chi,p,v)V^{\sharp}(t,\chi)\right) \ \leq \ \frac{\partial V^{\sharp}(t,\chi)}{\partial t}$$

The valuation function $V^{\sharp}$ is actually a solution to the nonlinear Hamilton-Jacobi-Isaacs partial differential equation

$$-\frac{\partial \mathbf{v}(t,\chi)}{\partial t}+\inf_{p\in P(\chi)}\sup_{v\in Q(\chi)}\left(\sum_{i=1}^{n}\left(\frac{\partial \mathbf{v}(t,\chi)}{\partial\chi_i}-p_i\right)f_i(\chi,v)+\mathbf{m}(\chi,p,v)\mathbf{v}(t,\chi)\right) \ = \ 0$$

satisfying the initial condition

$$\mathbf{v}(0,\chi) \ = \ \mathbf{u}(\chi)$$

on the subset

$$\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v}) \ := \ \{(t,\chi) \mid \mathbf{c}(t,\chi) \ > \ \mathbf{v}(t,\chi) \ \geq \ \mathbf{v}(t,\chi)\}$$

For each of the game rules (4), these subsets are written

1. **prescribed end rule**:

$$\Omega_{(\mathbf{0},\mathbf{u}_\infty)}(\mathbf{v}) \ := \ \{(t,\chi) \mid t>0 \ \& \ \mathbf{v}(t,\chi) \ \geq \ 0\}$$

2. **time span rule**

$$\Omega_{(\mathbf{u},\mathbf{u}_\infty)}(\mathbf{v}) \ := \ \{(t,\chi) \mid t>0 \ \& \ \mathbf{v}(t,\chi) \ \geq \ \mathbf{u}(\chi)\}$$

3. **first winning time rule**

$$\Omega_{(\mathbf{0},\mathbf{u})}(\mathbf{v}) \;:=\; \{(t,\chi) \mid t>0 \;\&\; \mathbf{u}(\chi) \;>\; \mathbf{v}(t,\chi) \;\geq\; 0\}$$

Actually, the solution of the above partial differential equation is taken in the "contingent sense", where the directional derivatives are the contingent epiderivatives $D_{\uparrow}\mathbf{v}(t,\chi)$ of $\mathbf{v}$ at (t,χ). They are defined by

$$D_{\uparrow}\mathbf{v}(t,\chi)(\lambda,v) \;:=\; \liminf_{h\to 0+,\, u\to v} \frac{\mathbf{v}(t+h\lambda,\chi+hu)}{h}$$

(see for instance [25, Aubin & Frankowska] and [63, Rockafellar & Wets]).

Definition 5.2 (Dynamical Core) *Consider the dynamic fuzzy cooperative game with game rules (5). The dynamical core* Γ *of the corresponding fuzzy dynamical cooperative game is equal to*

$$\begin{cases} \Gamma(t,\chi) \;:=\; \{p \in P(\chi) \;\; \textit{such that} \\ \sup_{v\in Q(\chi)}(D_{\uparrow}V^{\sharp}(t,\chi)(-1,f(\chi,v)) - \langle p, f(\chi,v)\rangle + \mathbf{m}(\chi,p,v)V^{\sharp}(t,\chi)) \;\leq\; 0\} \end{cases}$$

where $V^{\sharp}$ *is the corresponding value function.*

We can prove that for each feedback $\widetilde{p}(t,\chi) \in \Gamma(t,\chi)$ being a selection of the dynamical core Γ, all evolutions $(\chi(\cdot),v(\cdot))$ of the system

$$\begin{cases} i) & \chi'(t) \;=\; f(\chi(t),v(t)) \\ ii) & y'(t) \;=\; \langle \widetilde{p}(T-t,\chi(t)),\chi(t)\rangle - \mathbf{m}(\chi(t),\widetilde{p}(T-t,\chi(t))y(t) \\ iii) & v(t) \;\in\; Q(\chi(t)) \end{cases} \tag{11}$$

satisfy the corresponding condition (5).

5.7 The Static Case as Infinite Versatility

Let us consider the case when $\mathbf{m}(\chi,p,v)=0$ (self-financing of fuzzy coalitions) and when the evolution of coalitions is governed by $f(\chi,v)=v$ and $Q(\chi)=rB$. Then the dynamical core is the subset $\Gamma(t,\chi)$ of imputations $p\in P(\chi)$ satisfying on $\Omega(V^{\sharp})$ the equation[9]

$$r\left\|\frac{\partial V^{\sharp}(t,\chi)}{\partial \chi} - p\right\| \;=\; \frac{\partial V^{\sharp}(t,\chi)}{\partial t}$$

---

[9] when $p=0$, we find the eikonal equation.

Now, assuming that the data and the solution are smooth we deduce formally that letting the versatility $r \to \infty$, we obtain as a limiting case that $p = \frac{\partial V^\sharp(t,\chi)}{\partial \chi}$ and that $\frac{\partial V^\sharp(t,\chi)}{\partial t} = 0$. Since $V^\sharp(0,\chi) = \mathbf{u}(\chi)$, we infer that in this case $\Gamma(t,\chi) = \frac{\partial \mathbf{u}(\chi)}{\partial \chi}$, i.e., the Shapley value of the fuzzy static cooperative game when the characteristic function $\mathbf{u}$ is differentiable and positively homogenous, and the core of the fuzzy static cooperative game when the characteristic function $\mathbf{u}$ is concave, continuous and positively homogenous. ■

6 The Viability/Capturability Strategy

6.1 The Epigraphical Approach

The *epigraph* of an extended function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty$ is defined by

$$\mathcal{E}p(\mathbf{v}) \ := \ \{(\chi,\lambda) \in X \times \mathbf{R} \mid \mathbf{v}(\chi) \le \lambda\}$$

We recall that *an extended function* $\mathbf{v}$ *is convex (resp. positively homogeneous) if and only if its epigraph is convex (resp. a cone)* and that the epigraph of $\mathbf{v}$ is closed if and only if $\mathbf{v}$ is lower semicontinuous:

$$\forall\, \chi \in X, \ \ \mathbf{v}(\chi) \ = \ \liminf_{y \to x} \mathbf{v}(y)$$

With these definitions, we can translate the viability/capturability conditions (5) in the following geometric form:

$$\left\{ \begin{array}{ll} i) & \forall\, t \in [0,t^*], \ \ (T-t, \chi(t), y(t)) \in \mathcal{E}p(\mathbf{b}) \\ & \text{(viability constraint)} \\ ii) & (T-t^*, \chi(t^*), y(t^*)) \in \mathcal{E}p(\mathbf{c}) \\ & \text{(capturability of a target)} \end{array} \right. \tag{12}$$

This “epigraphical approach” proposed by J.-J. Moreau and R.T. Rockafellar in convex analysis in the early 60’s [10], has been used in optimal control by H. Frankowska in a series of papers [51, 52, 53, Frankowska] and [26, Aubin & Frankowska] for studying the value function of optimal control problems and characterize it as generalized solution (episolutions and/or viscosity solutions) of (first-order) Hamilton-Jacobi-Bellman equations, in [18, 22, 14, 13, Aubin] for characterizing and constructing Lyapunov functions, in [40, 41, 42, 43, Cardaliaguet] for characterizing the minimal time function, in [61, Pujal] and [28, Aubin, Pujal & Saint-Pierre] in finance and other authors since. This is this approach that we adopt and adapt here, since the concepts of “capturability of a target” and of “viability” of a constrained set allows us to study this problem under a new light (see for instance [13,

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[10] see for instance [25, Aubin & Frankowska] and [63, Rockafellar & Wets] among many other references.

Aubin] and [9, Aubin] for economic applications) for studying the evolution of the state of a tychastic control system subjected to viability constraints in control theory and in dynamical games against nature or robust control (see [62, Quincampoix], [40, 41, 42, 43, Cardaliaguet], [44, Cardaliaguet, Quincampoix & Saint-Pierre]. Numerical algorithms for finding viability kernels have been designed in [64, Saint-Pierre] and adapted to our type of problems in [61, Pujal].

The properties and characterizations of the valuation function are thus derived from the ones of guaranteed viable-capture basins, that are easier to study — and that have been studied — in the framework of plain constrained sets K and targets $C \subset K$ (see [2, 3, Aubin] and [21, Aubin & Catté] for recent results on that topic).

6.2 Introducing Auxiliary Dynamical Games

We observe that the evolution of $(T-t, \chi(t), y(t))$ made up of the backward time $\tau(t) := T-t$, of fuzzy coalitions $\chi(t)$ of the players, of imputations and of the payoff $y(t)$ is governed by the dynamical game

$$\left\{\begin{array}{ll} i) & \tau'(t) \;=\; -1 \\ ii) & \forall\, i = 0, \ldots, n, \;\; \chi_i'(t) \;=\; f_i(\chi(t), v(t)) \\ iii) & y'(t) \;=\; -y(t)\mathbf{m}(\chi(t), p(t), v(t)) + \langle p(t), f(\chi(t), v(t))\rangle \\ & \text{where } p(t) \;\in\; P(\chi(t)) \;\&\; v(t) \;\in\; Q(\chi(t)) \end{array}\right. \tag{13}$$

starting at (T, χ, y). We summarize it in the form of the dynamical game

$$\left\{\begin{array}{ll} i) & z'(t) \;\in\; g(z(t), u(t), v(t)) \\ ii) & u(t) \;\in\; P(z(t)) \;\&\; v(t) \in Q(z(t)) \end{array}\right.$$

where $z := (\tau, \chi, y) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}$, where the controls $u := p$ are the imputations, where the map $g : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \rightsquigarrow \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ is defined by $g(z, v)$

$$= (-1, f(\chi, v), u, -\mathbf{m}(\chi, u, v)y + \langle u, f(\chi, v)\rangle)$$

where u ranges over $P(z) := P(\chi)$ and v over $Q(z) := Q(\chi)$.

We say that a selection $z \mapsto \widetilde{p}(z) \in P(z)$ is a feedback, regarded as a strategy. One associates with such a feedback chosen by the decision maker or the player the evolutions governed by the tychastic differential equation

$$z'(t) \;=\; g(z(t), \widetilde{p}(z(t)), v(t))$$

starting at time 0 at z.

6.3 Introducing Guaranteed Capture Basins

We now define the guaranteed viable-capture basin that are involved in the definition of guaranteed valuation subsets.

Definition 6.1 *Let K and $C \subset K$ be two subsets of Z.*

The guaranteed viable-capture basin *of the target C viable in K is the set of elements $z \in K$ such that there exists a continuous feedback $\widetilde{p}(z) \in P(z)$ such that for every $v(\cdot) \in Q(z(\cdot))$, for every solutions $z(\cdot)$ to $z' = g(z, \widetilde{p}(z), v)$, there exists $t^* \in \mathbf{R}_+$ such that the viability/capturability conditions*

$$\begin{cases} i) & \forall\, t \in [0, t^*], \quad z(t) \in K \\ ii) & \qquad\qquad\quad\; z(t^*) \ \in \ C \end{cases}$$

are satisfied.

6.4 The Strategy

We thus observe that

Proposition 6.2 *The guaranteed valuation subset $\mathcal{V}^\sharp$ is the guaranteed viable-capture basin under the dynamical game (13) of the epigraph of the function $\mathbf{c}$ viable in the epigraph of the function $\mathbf{b}$.*

Since we have related the guaranteed valuation problem to the much simpler — although more abstract — study of guaranteed viable-capture basin of a target and other guaranteed viability/capturability issues for dynamical games,

1. we first "solve" these "viability/capturability problems" for dynamical games at this general level, and in particular, study the tangential conditions enjoyed by the guaranteed viable-capture basins,
2. and use set-valued analysis and nonsmooth analysis for translating the general results of viability theory to the corresponding results of the auxiliary dynamical game, in particular translating tangential conditions to give a meaning to the concept of a generalized solution (Frankowska's episolutions or, by duality, viscosity solutions) to Hamilton-Jacobi-Isaacs variational inequalities.

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