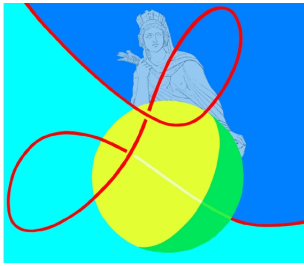


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Evolutions of Societies and Cultural Codes

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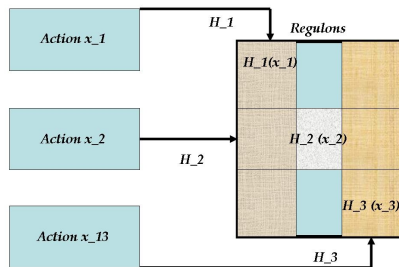


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1 *“If no one asks me, I know : if I wish to explain it to one that asketh, I know not : yet I say boldly that I know, ...”*

confessed Saint Augustine, in book XI (11.14.17) of his *Confessions*.



A definition ?

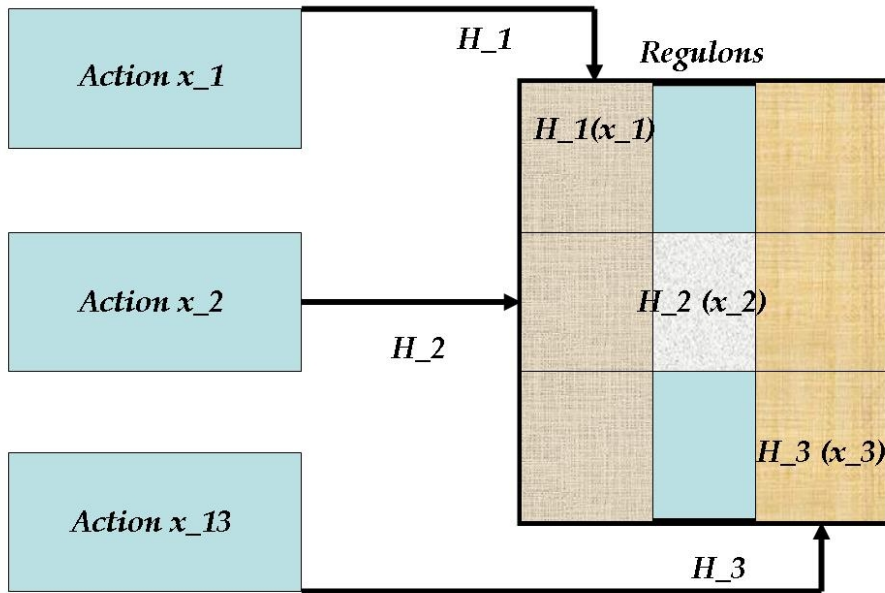
We are brains which perceive the outside world and which intercommunicate in various ways, using natural language, mathematics, bodily expressions, pictorial and musical techniques, etc.

It is the consensus on the consistency of individual perceptions of the environment, which in some way measures the degree of reality in a given social group.

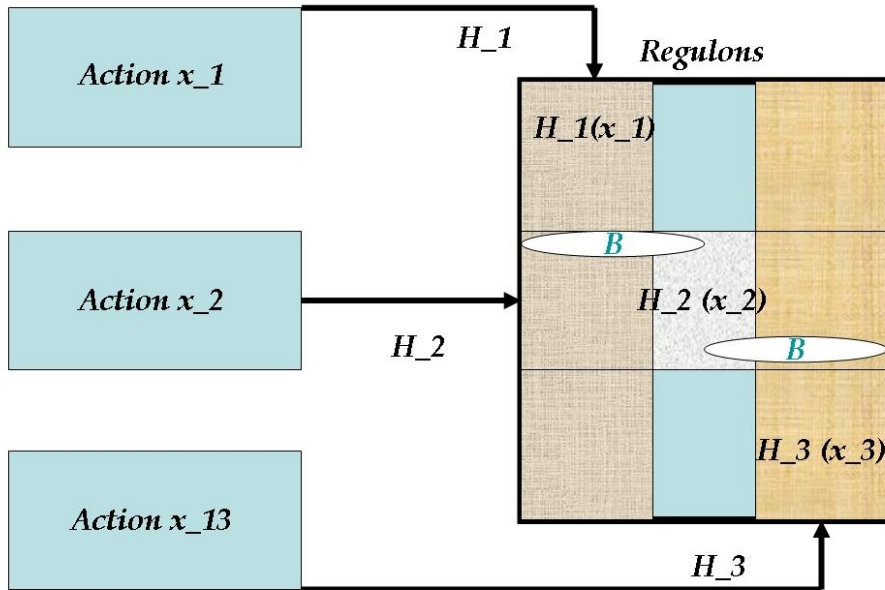
Since our brains were built on the same model, and since the ability to believe in explanations appears to be innate and universal, there is a very good chance that a social group may have a sufficiently broad consensus that its members share a common concept of reality.

But prophets and dissidents often challenge this consensus, while high priests and guardians of the ideology tend to dogmatize it and impose it on the members of the social group. (Moreover, quite often prophets and sages themselves become the high priests and guardians of the ideology, **and never the other way around.**)

2 Cultural Regulons and Societies



Let n agents in $N := \{i = 1, \dots, n\}$ choose actions $x_i \in X^i$ in spaces X_i . We introduce a space $U := \mathbb{R}^m$ of *cultural regulons*. The *culture* is a given subset $B \subset U$ of cultural regulons. The *cultural knowledge* of each agent $i \in N$ is a set-valued map $H_i : X_i \rightsquigarrow U$ associating with any action $x_i \in X_i$ a subset $H_i(x_i) \subset U$ of cultural regulons.



A **culture** $B \subset U$ is said a **consensus** on a collective action $x \in X^S$ of a coalition $S \subset N$ of agents if

$$\forall i \in S, B \subset H_i(x_i)$$

The **society** $S_H(B, x)$ associated with a culture $B \subset U$ and a collective action $x \in X^S$ is the coalition of agents $i \in N$ such that $B \subset H_i(x_i)$. The culture B is the **"reality"** perceived by agents of the society $S_H(B, x)$.

The action space $X_i := \mathbb{R}^l \times \mathbb{R}$ are made of commodities $x_i \in \mathbb{R}^l$ and incomes $r_i \in \mathbb{R}$. We choose as regulons the prices p ranging over the space $U := \mathbb{R}^{l^*}$. The map $H_i : \mathbb{R}^l \times \mathbb{R} \rightsquigarrow \mathbb{R}^{l^*}$ describing budgetary constraints are defined by

$$H_i(x, r) := \{p \in \mathbb{R}^{l^*} \text{ such that } \langle p, x \rangle \leq r\}$$

A subset $B \subset \mathbb{R}^{l^*}$ of prices is a consensus on a collective behavior $(x_i, r_i)_{i \in S}$ of actions $(x_i, r_i) \in \mathbb{R}^l \times \mathbb{R}$ taken by agents of the coalition S if

$$\forall p \in B, \forall i \in S, \langle p, x_i \rangle \leq r_i$$

This means that each agent can respect their budgetary constraints using the same prices $p \in B$.

The *society* $S_H(B, (x_i, r_i)_{i \in S})$ associated with a subset $B \subset \mathbb{R}^{l^*}$ of prices and a collective action $(x_i, r_i)_{i \in S}$ is the coalition of agents $i \in N$ such that

$$\forall p \in B, \langle p, x_i \rangle \leq r_i$$

Consider an input space P and an output vector space Q , with which we associate the product $X_i := P \times Q$ regarded as a set of actions (x_i, y_i) taken by agent i . We choose as regulons the synaptic matrices W ranging over the space $U := \mathcal{L}(P, Q)$. The map $H_i : P \times Q \rightsquigarrow \mathcal{L}(P, Q)$ describing neural networks are defined by

$$H_i(x, y) := \{W \in \mathcal{L}(P, Q) \text{ such that } y = g_i(Wx)\}$$

A subset $B \subset \mathcal{L}(P, Q)$ of synaptic matrices is a consensus on a collective behavior $(x_i, y_i)_{i \in S}$ of actions $(x_i, y_i) \in P \times Q$ taken by agents of the coalition S if they implement their actions (mapping x_i to y_i) using the same synaptic matrices $W \in B$:

$$\forall W \in B, \forall i \in S, y_i = g_i(Wx_i)$$

The **society** $S_H(B, (x_i, y_i)_{i \in S})$ associated with a subset $B \subset \mathcal{L}(P, Q)$ and a collective action $(x_i, y_i)_{i \in S}$ is the coalition of agents $i \in N$ such that

$$\forall W \in B, y_i = g_i(Wx_i)$$

We associate with any coalition $S \subset N$ the subspace

$$X^S := \prod_{i \in S} X_i$$

In particular, $X^N := \prod_{i=1}^n X_i$, $X^i = X_i$ and $X^\emptyset := \{0\}$.

We interpret the set-valued maps $\widehat{H}_S : X^S \rightsquigarrow U$ associating with any actions $x_i \in X_i$ of the agents belonging to the coalition S the subset

$$\widehat{H}_S(x) := \bigcap_{i \in S} H_i(x_i)$$

as the *cultural map* of the coalition S .

It is the cultural knowledge needed to implement a collective action of coalition S .

The *inverse focus* $\widehat{H}_S^{-1}(B) \subset S$ of a culture by a coalition $S \subset N$ is the subset

$$\widehat{H}_S^{-1}(B) := \{x \in X^S \text{ such that } \forall i \in S, B \subset H_i(x_i)\}$$

or

$$\widehat{H}_S^{-1}(B) := \{x \in X^S \text{ such that } B \subset \widehat{H}_S(x)\}$$

Consequently, *a culture B is a consensus on a collective action $x \in X^S$ for a coalition S of agents if and only if $x \in \widehat{H}_S^{-1}(B)$, or, equivalently, if and only if the coalition S is contained in the society $S_H(B, x)$.*

Let $F : X \rightsquigarrow Y$ be a set-valued map. When $A \subset X$, the subset $F(A)$ is defined by

$$F(A) := \bigcup_{x \in A} F(x)$$

is called the *image of A by F* and

$$\widehat{F}(A) = \bigcap_{x \in A} F(x) = \{y \mid A \subset F^{-1}(y)\} \subset F(A)$$

is called the *focus of A by F* .

They are two different extensions of F as maps from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$.

When $B \subset Y$, the subset

1. $F^{-1}(B)$ is defined by

$$F^{-1}(B) := \{x \mid F(x) \cap B \neq \emptyset\}$$

is called the *inverse image* of B by F ,

2. $F^{\ominus 1}(B)$ defined by

$$F^{\ominus 1}(B) := \{x \mid F(x) \subset B\}$$

is called the *core* of B by F ,

3. $\widehat{F^{-1}}(B)$ defined by

$$\widehat{F^{-1}}(B) := \bigcap_{y \in B} F^{-1}(y) = \{x \mid B \subset F(x)\} \subset F^{-1}(B)$$

is called the *inverse focus* of B by F .

These maps provide three classes of examples of viability constraints. We shall focus ... on inverse focuses $K := \widehat{F^{-1}}(B)$.

1. The focus and inverse focus are decreasing :

If $A_1 \subset A_2$, then $\widehat{F}(A_2) \subset \widehat{F}(A_1)$ and if $B_1 \subset B_2$, then $\widehat{F}^{-1}(B_2) \subset \widehat{F}^{-1}(B_1)$.

2. The focus and inverse focus transform unions into intersections :

$$\widehat{F} \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} \widehat{F}(A_i) \quad \& \quad \widehat{F}^{-1} \left(\bigcup_{i \in I} B_i \right) = \bigcap_{i \in I} \widehat{F}^{-1}(B_i)$$

1. The following statements are equivalent :

$$\left\{ \begin{array}{l} (i) \quad A \subset \widehat{F^{-1}}(B) \\ (ii) \quad B \subset \widehat{F}(A) \\ (iii) \quad A \times B \subset \text{Graph}(F) \end{array} \right. \quad (1)$$

2. The maps $\widehat{F^{-1}}\widehat{F}$ and $\widehat{F}\widehat{F^{-1}}$ are *closings*.

3. A subset $A \times B \subset \text{Graph}(F)$ is *maximal* in the graph of F if and only if $A := \widehat{F^{-1}}(B)$ and $B = \widehat{F}\widehat{F^{-1}}(B)$ or, equivalently, if $A := \widehat{F^{-1}}(\widehat{F}(A))$ and $B := \widehat{F}(A)$.

A subset $\text{Graph}(F) \subset \prod_{i \in I} X_i$ is regarded as the graph of a relation F between elements $(x_i)_{i \in I}$ of the spaces X_i .

The graph $\text{Graph}(\widehat{F})$ of the *hyperrelation* of a relation $\text{Graph}(F)$ is made of families $(A_i)_{i \in I}$, called *factorizations of F* , such that $\prod_{i \in I} A_i \subset \text{Graph}(F)$

$$\text{Graph}(\widehat{F}) = \left\{ (A_i)_{i \in I} \text{ such that } \prod_{i \in I} A_i \subset \text{Graph}(F) \right\}$$

We are naturally interested in *maximal factorizations* $(A_i)_{i \in I}$ of $\text{Graph}(F)$ such that $\prod_{i \in I} A_i$ is maximal.

Given *mutational equations* governing the evolution of subsets $A_i(t) \subset X_i$, study the *viability of factorizations* of a relation $\text{Graph}(H)$:

$$\forall t \geq 0, \prod_{i \in I} A_i(t) \subset \text{Graph}(H)$$

Simple example : Given Lipschitz set-valued maps with compact values $F_i : X_i \rightsquigarrow X_i$ such that the graph $\text{Graph}(H)$ of the relation H is invariant under $\prod_{i \in I} F_i : \prod_{i \in I} X_i \rightsquigarrow \prod_{i \in I} X_i$ defined by $\left(\prod_{i \in I} F_i \right)(x) := \prod_{i \in I} F_i(x_i)$, *factorizations of H are viable* :

$$\prod_{i \in I} \vartheta_{F_i}(t, A_i) \subset \text{Graph}(H)$$



4 Viability of Inverse Focus Constraints

Let us consider a set-valued map $H : X \rightsquigarrow U$. When $B \subset U$, the viability of inverse focus constraints requires that at each instant, the subset \mathbf{B} is a subset of $H(x(t))$:

$$\forall t \geq 0, \mathbf{B} \subset H(x(t)) \text{ or, equivalently } x(t) \in K := \widehat{H^{-1}}(\mathbf{B}) \quad (2)$$

We begin by studying the viability issues of such constraints when the evolution $t \mapsto x(t) \in X$ governed by a differential equation $x'(t) = f(x(t))$.

Pierre de Fermat (1637) : The graph of the adjacent derivative $D^b H(x, u)$ of H at $(x, u) \in \text{Graph}(H)$ is the adjacent cone to the graph of H at (x, u) :

$$\text{Graph}(D^b H(x, u)) := T_{\text{Graph}(H)}^b(x, u) := \text{Liminf}_{h \rightarrow 0} \frac{\text{Graph}(H) - (x, u)}{h}$$

Setting $\nabla_h H(x, u)(x') := \frac{H(x + hx') - u}{h}$, we obtain Gottfried Leibniz's version

$$\text{Graph}(D^b H(x, u)) = \text{Liminf}_{h \rightarrow 0} \text{Graph}(\nabla_h H(x, u))$$

Consequence :

$$\bigcap_{u \in B} D^b H(x, u)^{-1}(0) \subset T_{\widehat{H^{-1}(B)}}(x)$$



Let us assume that

$$\forall x \in \widehat{H^{-1}(B)}, f(x) \in \bigcap_{u \in B} D^b H(x, u)^{-1}(0)$$

Therefore, the subset $\widehat{H^{-1}(B)}$ is viable under the differential equation $x'(t) = f(x(t))$.

If the inverse focus viability constraints are not viable under a differential equation $x'(t) = f(x(t))$, the dynamics of the actions can then be corrected by modifying it with *viability multipliers* $p(\cdot)$. We set

$$P(x, B) := f(x) - \bigcap_{u \in B} D^b \widehat{H}(x, u)^{-1}(0)$$

Therefore, for any continuous selection $x \mapsto \tilde{p}(x)$ of the set-valued map $x \rightsquigarrow P(x, B)$, the sociability constraints **2** are viable under the corrected joint evolutionary-morphological system

$$x'(t) = f(x(t)) - \tilde{p}(x(t))$$

We study the evolution $t \mapsto x(t) \in X$ of states regulated by subsets $\mathbf{B}(t) \subset U$ by a parameterized differential equation

$$x'(t) = f(x(t), \mathbf{B}(t))$$

Let us assume further that the set of regulons evolves under *mutational equation*

$$\dot{\mathbf{B}}(t) \ni G(x(t), \mathbf{B}(t))$$

where the right hand side $G(x, B)(\cdot) \in \text{LIP}(U, U)$.

The *joint evolutionary-morphological system*

$$\begin{cases} (i) & x'(t) = f(x(t), \mathbf{B}(t)) \\ (ii) & \dot{\mathbf{B}}(t) \ni G(x(t), \mathbf{B}(t)) \end{cases} \quad (3)$$

governs the joint evolution of the state and of the set of regulons.

The viability of inverse focus constraints requires that at each instant the subset $\mathbf{B}(t)$ is a subset of $H(x(t))$:

$$\forall t \geq 0, \mathbf{B}(t) \subset H(x(t)) \text{ or } x(t) \in \mathbf{K}(t) := \widehat{H}^{-1}(\mathbf{B}(t)) \quad (4)$$

If

$$\forall B \in \mathcal{K}(U), \forall x \in \widehat{H}^{-1}(B), f(x, B) \in \bigcap_{u \in B} D^b \widehat{H}(x, u)^{-1}(G(x, B)(u))$$

then from any initial set B of regulons and for any state $x \in \widehat{H}^{-1}(B)$, there exists a joint evolution $(x(t), \mathbf{B}(t))$ governed by the joint evolutionary-morphological system (3) such that, at each instant, $x(t)$ belongs to $\widehat{H}^{-1}(\mathbf{B}(t))$ associated with the set $\mathbf{B}(t)$ of regulons.

One can take into account that the map $t \mapsto H(t)$ where $H(t) : X \rightsquigarrow U$ evolves and is governed by a mutational equation of the form

$$\overbrace{\text{Graph}(H(t))}^{\circ} \ni \Omega \times \Psi$$

governing evolutions

$$H(t; x) = \vartheta_{\Psi}(t, H(\vartheta_{-\Omega}(t, x)))$$

If $\forall H, \forall (x, B)$ such that $\{x\} \times B \subset \text{Graph}(H)$

$$f(x, B) \in \bigcap_{u \in B} \left(\bigcup_{z \in \Psi(u)} D^b \widehat{H}(x, u)^{-1}(\Phi(u) - z) \right) + \Omega(x)$$

then, from any cultural map $H : X \rightsquigarrow U$ with compact graph, any subset $B \in \mathcal{K}(U)$ of regulons and any $x \in \widehat{H}^{-1}(B)$, there exists an evolution $x(\cdot)$ governed by differential equation $x'(t) = f(x(t), \vartheta_{\Phi}(t, B))$ starting from x and satisfying

$$\forall t \geq 0, \vartheta_{\Phi}(t, B) \subset \vartheta_{\Psi}(t, H(\vartheta_{-\Omega}(t, x(t))))$$

The inertia of a tube $\mathbf{B}(\cdot)$ on an interval $[0, T[$ is defined by

$$\sup_{t \in [0, T[} \|\overset{\circ}{\mathbf{B}}(t)\|_{\Lambda}$$

and the inertia function α is defined by

$$\alpha(T, x, B) := \inf \sup_{t \in [0, T[} \|\overset{\circ}{\mathbf{B}}(t)\|_{\Lambda}$$

The epigraph of the inertia function α is the confinement basin of $\{0\} \times \text{Graph}(\mathbf{H}) \times \mathbb{R}_+$ viable in $\mathbb{R}_+ \times \text{Graph}(\mathbf{H}) \times \mathbb{R}_+$ under the auxiliary joint evolutionary-morphological system

$$\left\{ \begin{array}{l} (i) \quad \tau'(t) = -1 \\ (ii) \quad x'(t) = f(x(t), \mathbf{B}(t)) \\ (iii) \quad \overset{\circ}{\mathbf{B}}(t) \ni \Phi(t) \\ (iv) \quad y'(t) = 0 \text{ where } \|\Phi(t)\|_{\Lambda} \leq y(t) \end{array} \right.$$

Consider three spaces X , Y and U . The graph of a triad H is a subset $\text{Graph}(H) \subset X \times Y \times U$.

It can be regarded as the graph of map $H : X \times Y \rightsquigarrow U$, or a map $\Phi : X \times U \rightsquigarrow Y$ defined by

$$H(x, y) := \{u \in U \text{ such that } (x, y, u) \in \text{Graph}(H)\}$$

and

$$\Phi(x, u) := \{y \in Y \text{ such that } (x, y, u) \in \text{Graph}(H)\}$$

We observe that

$$H^{-1}(u) = \text{Graph}(\Phi(\cdot, u))$$

is the graph of the elementary set-valued map $\Phi(\cdot, u)$.

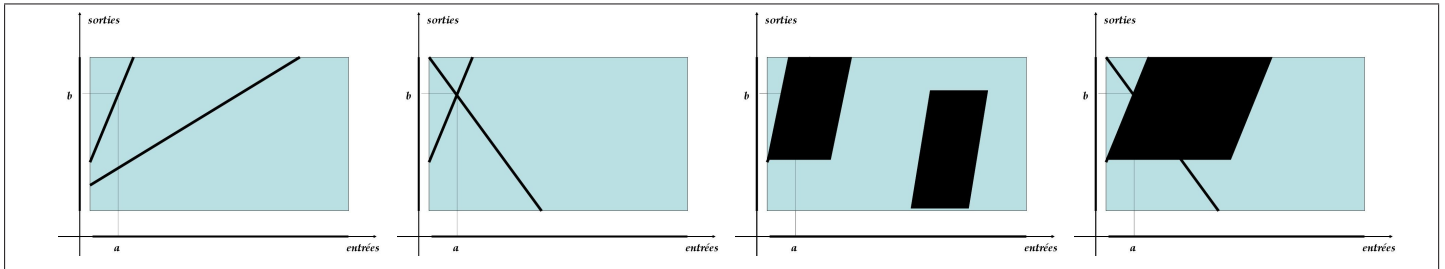


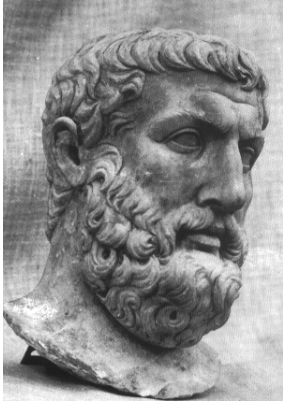
Figure 5.1 Multivocality of maps and of the triad *The subset B is made of two elements. Four situations are possible : The maps $\Phi(\cdot, u)$ and H are single-valued, the maps $\Phi(\cdot, u)$ are single-valued and H is set-valued, the maps $\Phi(\cdot, u)$ are set-valued and H is single-valued, the maps $\Phi(\cdot, u)$ and H are set-valued.*



We associate with any map $\Phi : X \rightsquigarrow Y$ its focus

$$\widehat{H}(\Phi) := \widehat{H}(\text{Graph}(\Phi)) := \bigcap_{(x,y) \in \text{Graph}(\Phi)} H(x,y) \subset U$$

It can be regarded as an *abstraction* of the set-valued map Φ , the subset of regulons recognized by *all input-output nodes* $(x,y) \in \text{Graph}(\Phi)$.



2 Abstraction. *One meaning of abstraction is the quest of sub-concepts common to a giving concept. It consists in throwing away irrelevant aspects, a kind of **intellectual striptease up to a bare minimum**, the modern version of what Parmenides³ and the pre-Socratic Greeks called **aletheia**, the dis-covering, un-veiling of the world that surrounds us.*

Is this drive to “abstraction”, isolating, in a given perspective, the relevant information in each concept and investigate the interplay between them, the only advance of humanity ?



The inverse focus $\widehat{H^{-1}(B)} = \text{Graph}(\widehat{\Phi(\cdot, B)})$ of a subset $B \subset U$ is the graph of the set-valued map $\widehat{\Phi(\cdot, B)}$ defined by

$$\widehat{\Phi(x, B)} = \bigcap_{u \in B} \Phi(x, u)$$

One can regard the closing $\widehat{H^{-1}\widehat{H}}$ as the *generalization operator* of any map $\Phi : X \rightsquigarrow Y$ by the map $\widehat{\Phi(\cdot, \widehat{H}(\Phi))}$, since $\text{Graph}(\widehat{\Phi(\cdot, \widehat{H}(\Phi))}) = \widehat{H^{-1}\widehat{H}(\Phi)}$ and the closing $\widehat{H}(\widehat{H^{-1}(B)})$ as the *saturation of B*.

Consider a training set $\text{Graph}(\Phi) \subset P \times Q$ and choose as regulons the space $U := \mathcal{L}(P, Q)$ of synaptic matrices W . The map $H : P \times Q \rightsquigarrow \mathcal{L}(P, Q)$ defined by

$$H(x, y) := \{W \in \mathcal{L}(P, Q) \text{ such that } y = g(Wx)\}$$

The focus $\widehat{H}(\Phi)$ is the set of synaptic matrices W which have "learned" the training set and $\widehat{H}^{-1}\widehat{H}(\Phi)$ is the generalization of the training set $\text{Graph}(\Phi)$.

Heavy Learning Algorithm : Let $s : U \in \mathcal{P}(U) \times U$ be a map satisfying

$$\forall u \in U, \forall B \subset U, s(u, B) \in B \text{ and } \forall u \in B, s(u, B) = u$$

Starting with $(x_1, y_1) \in \text{Graph}(\Phi)$ and $u_1 \in H(x_1, y_1)$, we associate with $(x_2, y_2) \in \text{Graph}(\Phi)$ the regulon $u_2 := s(u_1, \widehat{H}((x_1, y_1) \cup (x_2, y_2)))$, and recursively,

$$u_n := s(u_{n-1}, \widehat{H}((x_1, y_1) \cup (x_2, y_2), \dots \cup (x_n, y_n)))$$

A partition $P : I \rightsquigarrow I$ of the subset I of indexes is a set-valued map such that $I = \bigcup_{j \in I} P(j)$ and that for any pair $j^1 \neq j^2$, $P(j^1) \cap P(j^2) = \emptyset$. The finest partition $P^\sharp : I \rightsquigarrow I$ is defined by $P^\sharp(i) := \{i\}$.

In order to refine the partition P , regarded as the partition of level 1, we associate with any $j^1 \in \text{Dom}(P)$ a partition $P^{j^1} : P(j^1) \rightsquigarrow P(j^1)$ associating with any $j^2 \in \text{Dom}(P^{j^1})$ a subset $P^{j^1}(j^2) \subset P(j^1)$. We thus infer that

$$I = \bigcup_{j^1 \in \text{Dom}(P)} \bigcup_{j^2 \in \text{Dom}(P^{j^1})} P^{j^1}(j^2)$$

A sequence of partitions P^{j^1, j^2, \dots, j^k} at the k th level of a cascade of refined partitions are defined recursively as partitions

$$P^{j^1, j^2, \dots, j^k} : P^{j^1, j^2, \dots, j^{k-1}}(j^k) \rightsquigarrow P^{j^1, j^2, \dots, j^{k-1}}(j^k)$$

Then

$$I = \bigcup_{j^1 \in \text{Dom}(P)} \bigcup_{j^2 \in \text{Dom}(P^{j^1})} \dots \bigcup_{j^k \in \text{Dom}(P^{j^1, j^2, \dots, j^{k-1}})} P^{j^1, j^2, \dots, j^{k-1}}(j^k)$$



If $P : I \rightsquigarrow I$ is a partition, we can write

$$A_I = \prod_{j \in \text{Dom}(P)} A_{P(j)} \text{ where } A_{P(i)} := \prod_{l \in P(i)} A_l$$

Refining one more step by taking partitions $P^{j^1} : P(j^i) \rightsquigarrow P(j^i)$, one can write

$$A_I = \prod_{j^1 \in \text{Dom}(P)} \prod_{j^2 \in \text{Dom}(P^{j^1})} A_{P^{j^1}(j^2)} \text{ where } A_{P^{j^1}(j^2)} = \prod_{j^3 \in P^{j^1}(j^2)} A_{j^3}$$

Therefore, we can associate with a *cascade of refined partitions* $P^{j^1, j^2, \dots, j^{k-1}}$ a decomposition of the product X_I in the following form

$$A_I = \prod_{j^1 \in \text{Dom}(P)} \prod_{j^2 \in \text{Dom}(P^{j^1})} \cdots \prod_{j^k \in \text{Dom}(P^{j^1, j^2, \dots, j^{k-1}})} A_{P^{j^1, j^2, \dots, j^{k-1}}(j^k)}$$

where

$$A_{P^{j^1, j^2, \dots, j^{k-1}}(j^k)} = \prod_{j^{k+1} \in P^{j^1, j^2, \dots, j^{k-1}}(j^k)} A_{j^{k+1}}$$

Let us consider a cascade of refined partitions of the set I of indexes and a relation $\mathcal{F} := \text{Graph}(F) \subset X_I$. A sequence of subsets $(A_i)_{i \in I}$ belongs to the finest factorization $\widehat{\mathcal{F}}$ of the relation \mathcal{F} if, at each level, there exists an element $\mathcal{F}_{P^{j^1, j^2, \dots, j^{l-1}}(j^l)}$ of the factorization $\widehat{\mathcal{F}_{P^{j^1, j^2, \dots, j^{l-1}}}}$ of the relation at level $l-1$ such that $\mathcal{F}_{P^{j^1, j^2, \dots, j^l}(j^{l+1})}$ is an element of the factorization $\widehat{\mathcal{F}_{P^{j^1, j^2, \dots, j^l}}}$ at level l .



*Merci pour votre attention
Thank you for your attention*



3 Fragments of Parmenides' poem *ALETHEIA*.

*“Come now, I will tell you - and bring away my story safely when you have heard it - the only ways of inquiry there are to think : the One, that it is and that it is not possible for it not to be, is the path of Persuasion (for it attends upon **Truth**), the other, that it is not and that it is necessary for it not to be, this I point out to you to be a path completely unlearnable, for neither may you know that which is not (for it is not to be accomplished) nor may you declare it. ”*