LASTRE

Laboratoire d'Applications des Systèmes Tychastiques Régulés

Bassins de connexion et stratégies Eupaliniennes

Séminaire Viabilité, Jeux, Contrôle

Jean-Pierre Aubin

IHP

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Contents

1 Introduction

Eupalinos, a Greek engineer, excavated around 550 BC a 1036 m. long tunnel 180 m. below Mount Kastro for building an aqueduct supplying Pythagoreion (then the capital of Samos) with water on orders of tyrant Polycrates. *He started to dig simultaneously the tunnel from both sides by two working teams who met in the center* of the channel and they had only 0,6 m. error. There is still no consensus on how he did it. However, this is the very *"Eupalinian strategy"* used ever since for building famous tunnels (under the Channel or the Mont-Blanc) or bridges: it consists in starting the construction at the same time from both end-points x and y and proceed until they collide, by continuously monitoring the progress of the construction.

Such models can also be used as mathematical metaphors in negotiation procedures when both actors start from opposite statements and try to reach a consensus by making mutual concessions step by step, continuously bridging the remaining gap.

Herodotus

Fig.13: The water duct where it exits the tunnel in the south

And about the Samians I have spoken at greater length, because they have three works which are greater than any others that have been made by Hellenes: first a passage beginning from below and open at both ends, dug through a mountain not less than a hundred and fifty fathoms [200 m] in height; the length of the passage is seven furlongs and the height and breadth each eight feet, and throughout the whole of it another passage has been dug twenty cubits in depth and three feet in breadth, through which the water is conducted and comes by the pipes to the city, brought from an abundant spring: and the designer of this work was a Megarian, Eupalinos the son of Naustrophos.

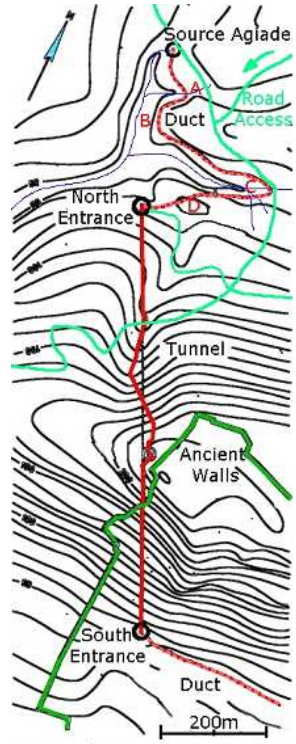

Fig.2: The situation according to Kienast, modified.

Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ the evolutionary system associated with the controlled system

$$x'(t) \;=\; f(x(t), u(t)) \text{ where } u(t) \;\in\; U(x(t))$$

and $K \subset X$ be an environment.

We denote by

$$\mathcal{S}^K(y, z) \;:=\; \text{Conn}_{\mathcal{S}}(K, (\{y\}, \{z\}))$$

the set of *Eupalinian evolutions* $x(\cdot)$ governed by the evolutionary system $\mathcal{S}$ viable in K connecting y to z, i.e., the set of evolutions $x(\cdot) \in \mathcal{S}(y)$ such that there exists a finite time $T \geq 0$ satisfying $x(T) = z$ and, for all $t \in [0, T]$, $x(t) \in K$.

The *Eupalinian kernel* $\mathcal{E} := \text{Eup}_{\mathcal{S}}(K) \subset K \times K$ is the subset of pairs (y, z) such that there exists at least one viable evolution $x(\cdot) \in \mathcal{S}^K(y)$ connecting y to z and viable in K.

Let us denote by $\text{Diag}(K) := \{(x,x)\}_{x\in K} \subset K \times K$ **the *diagonal of*** K**. The Eupalinian kernel** $\text{Eup}_{\mathcal{S}}(K)$ **of** K **under the evolutionary system** $\mathcal{S}$ **associated with the system**

$$x'(t) \;=\; f(x(t),u(t)) \text{ where } u(t) \;\in\; U(x(t))$$

is the capture basin

$$\text{Eup}_{\mathcal{S}}(K) \;=\; \text{Capt}_{(1)}(K\times K, \text{Diag}(K))$$

of the diagonal of K **viable in** $K \times K$ **under the auxiliary system**

$$\begin{cases} (i) & y'(t) \;=\; f(y(t),u(t)) \text{ where } u(t) \;\in\; U(y(t)) \\ (ii) & z'(t) \;=\; -f(z(t),z(t)) \text{ where } v(t) \;\in\; U(z(t)) \end{cases} \tag{1}$$

The *Eupalinian tube* $T \rightsquigarrow \mathrm{Eup}_{\mathcal{S}}(K)(T) \subset K \times K$ is the subset of pairs (y,z) such that there exists at least one evolution connecting y to z, viable in K and of duration smaller than or equal to T. The minimum duration of viable evolutions linking y to z is the *Eupalinian function*:

$$\epsilon_K(y,z) \; := \; \inf \left\{ T \text{ such that } \exists\, x(\cdot) \in \mathcal{S}^K(y) \textbf{ satisfying } x(T) = z \right\}$$

Let $B \subset K$ a source and $C \subset K$ be a target, the function

$$\epsilon_K(B,C) \; := \; \inf_{y \in B,\, z \in C} \epsilon_K(y,z)$$

is called the *Eupalinian distance* between the source B and the target C, i.e., the minimal duration of the viable evolutions connecting B to C.

Let us consider system

$$\left\{\begin{array}{lll} (i) & \tau'(t) = -1 & \\ (ii) & y'(t) \ = \ f(y(t), u(t)) \text{ where } u(t) \ \in \ U(y(t)) \\ (iii) & z'(t) \ = \ -f(z(t), z(t)) \text{ where } v(t) \ \in \ U(z(t)) \end{array}\right. \quad (2)$$

The graph of the Eupalinian tube $\text{Eup}_{\mathcal{S}}(K)(\cdot)$ **is the viable-capture basin of** $\{0\} \times \text{Diag}(K)$ **viable in** $\mathbb{R}_+ \times K \times K$ **under the system (2):**

$$\text{Graph}(T \rightsquigarrow \text{Eup}_{\mathcal{S}}(K)(2T)) \ = \ \text{Capt}_{(2)}(\mathbb{R}_+ \times K \times K, \{0\} \times \text{Diag}(K))$$

or, equivalently,

$$\epsilon_K(y, z) \ = \ 2 \inf_{(T,y,z) \in \text{Capt}_{(2)}(\mathbb{R}_+ \times K \times K, \{0\} \times \text{Diag}(K))} T$$

Let $(T, y, z) \in \mathrm{Capt}_{(2)}(\mathbb{R}_+ \times K \times K, \mathbb{R}_+ \times \mathrm{Diag}(K))$ **belong to the capture basin.**

This means that there is a forward evolution $\overrightarrow{y}(\cdot) \in \overrightarrow{\mathcal{S}}^K(y)$ **viable in** K**, a backward evolution** $\overleftarrow{z}(\cdot) \in \overleftarrow{\mathcal{S}}^K(z)$ **viable in** K **and a time** $t^\star \geq 0$ **such that, for all** $t \in [0, t^\star]$**,** $\overrightarrow{y}(t) \in K$**,** $\overleftarrow{z}(t) \in K$**,** $T - t^\star \geq 0$ **and** $\overrightarrow{y}(t^\star) = \overleftarrow{z}(t^\star)$**.**

In other words, $t^\star \in [0, T]$ **and that the evolution** $x(t)$ **defined by** $x(t) := \overrightarrow{y}(t)$ **for** $t \in [0, t^\star]$ **and** $x(t) := \overleftarrow{z}(2^{t^\star} - t)$ **for** $t \in [t^\star, 2t^\star]$ **is an evolution** $x(\cdot) \in \mathcal{S}(y)$ **governed by the differential inclusion starting at** y**, continuous at** $t^\star$ **because** $x(t^\star) = \overrightarrow{y}(t^\star) = \overleftarrow{z}(t^\star)$ **by construction which satisfies** $z(2t^\star) = \overleftarrow{z}(0) = y$**, and viable on** $[0, 2t^\star]$**.**

Therefore the pair (y, z) **belongs to the Eupalinian kernel** $\mathrm{Eup}_{\mathcal{S}}(K)(2T)$ **since** $t^\star \leq T$**.** ■

Viable evolutions connecting y to z in *minimal time* were called *brachistochrones*.

Their existence and computation was posed as a challenge by *Johann Bernoulli* in 1696, challenge met by *Gottfried Leibniz*, *Isaac Newton* (who solved it the very next day where is was presented to him), *Jacob Bernoulli*, *Gottfried Leibniz* and *Guillaume de L'Hopital* in a particular case.

Actually, Johann Bernoulli had originally found an incorrect proof that the curve is a cycloid, and challenged his brother Jakob to find the required curve. When Jakob correctly did so, Johann tried to substitute the proof for his own.

Let us consider the *minimal length function* $\gamma_K(x) : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\gamma_K(x) \ := \ \inf_{x(\cdot)\in\mathcal{S}^K(x)} \int_0^{\infty} \|x'(\tau)\| d\tau$$

The minimal connecting distance between y and z is defined by

$$\gamma_K(y,z) \ := \ \inf_{x(\cdot)\in\mathcal{S}^K(y) \ | \ x(T)=z} \int_0^{T} \|x'(\tau)\| d\tau$$

and the evolutions achieving the minimum are the *viable geodesics*. The *geodesic tube* is defined by

$$\begin{cases} \text{Geod}(K)(\gamma) \ := \\ \big\{(y,z) \in K \times K \text{ such that } x(\cdot) \in \mathcal{S}^K(y), \exists\, T \geq 0 \mid x(T) = z \\ \text{and } \displaystyle\int_0^T \|x'(\tau)\| d\tau \ \leq \ \gamma \Big\} \end{cases}$$

Let us consider the auxiliary control system

$$\left\{\begin{array}{lll} (i) & \lambda'(t) = -\|f(y(t),u(t))\| - \|f(z(t),v(t))\| \\ (ii) & y'(t) \ = \ f(y(t),u(t)) \text{ where } u(t) \ \in \ U(y(t)) \\ (iii) & z'(t) \ = \ -f(z(t),v(t)) \text{ where } v(t) \ \in \ U(z(t)) \end{array}\right. \quad (3)$$

The graph of the minimal connecting tube $\text{Geod}_{\mathcal{S}}(K)(\cdot)$ **is the viable-capture basin of** $\mathbb{R}_+ \times \text{Diag}(K)$ **viable in** $\mathbb{R}_+ \times K \times K$ **under the system (3):**

$$\forall\, T \geq 0, \ \ \text{Graph}(\text{Geod}(K)) \ = \ \text{Capt}_{(3)}(\mathbb{R}_+ \times K \times K, \{0\} \times \text{Diag}(K))(T)$$

Let $B \subset K$ be a subset regarded as a source, $C \subset K$ be a subset regarded as a target.

The *connection basin* is associated with the Eupalinian kernel by

$$\mathrm{Conn}_{\mathcal{S}}(K,(B,C)) := \bigcup_{(y,z)\in\mathcal{E}} \mathrm{Conn}_{\mathcal{S}}(K,(\{y\},\{z\}))$$

Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(-\infty, \infty; X)$ be an evolutionary system, $K \subset X$ be a environment, and $B \subset K$ be a source and $C \subset K$ be a target.

The connection basin is the largest subset $D \subset K$ of K that is connecting B to C *viable in* D.

The connection basin is the largest fixed point of the map $D \mapsto \mathrm{Conn}_{\mathcal{S}}(D, (B, C))$ contained in K.

Furthermore, all evolutions connecting B to C viable in K are actually viable in $\mathrm{Conn}_{\mathcal{S}}(D, (B, C))$.

The connection basin $\text{Conn}_{\mathcal{S}}(K, (B, C))$ **between a source** B **and a target** C **viable in the environment** K **is both forward and backward invariant relatively to** K**.**

A subset $D \subset K$ **is bilaterally invariant relatively to** K **if and only if** $\text{Conn}_{\mathcal{S}}(K, (D, D)) = D$**.**

4 Eupalinian Optimization

We introduce a cost function $\mathbf{c} : X \times X \mapsto \mathbb{R} \cup \{+\infty\}$ (regarded as a *connection cost*) and a Lagrangian $\mathbf{l} : (x,u) \rightsquigarrow \mathbf{l}(x,u)$.

We consider the Eupalinian optimization problem

$$\mathbf{U_c}(y,z) \ := \inf_{x(\cdot)\in\mathcal{S}^K(y,z),\, t^\star \geq 0 \,|\, x(2t^\star)=z} \left(\mathbf{c}(x(t^\star), x(t^\star)) + \int_0^{2t^\star} \mathbf{l}(x(t),u(t))dt \right)$$

- **By taking c $\equiv 0$ and l$(x,u) \equiv 1$, we find the problem of connecting two states by a viable evolution in minimal time (see the *brachistochrone* problem),**
- **By taking c $\equiv 0$ and l$(x,u) = \|f(x,u)\|$, we obtain the *viable geodesic* connecting two states by a viable evolution in minimal length,**
- **By taking c $\equiv 0$ and l$(x,u) = \varphi(x)$, we connect two states y and z by a viable evolution minimizing the occupational cost $\int_0^{2t^\star} \varphi(x(t))dt$, etc.**

Let us consider the auxiliary control system

$$\left\{ \begin{array}{lll} (i) & y'(t) \; = \; f(y(t), u(t)) \text{ where } u(t) \; \in \; U(y(t)) \\ (ii) & z'(t) \; = \; -f(z(t), v(t)) \text{ where } v(t) \; \in \; U(z(t)) \\ (iii) & \lambda'(t) = -\mathbf{l}(y(t), u(t)) - \mathbf{l}(z(t), v(t)) \end{array} \right. \tag{4}$$

Then

$$\mathbf{U_c}(y, z) \; = \; \inf_{(y,z,\lambda) \in \text{Capt}_{(4)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\text{Diag}(K) \times \mathbb{R}_+))} \lambda$$

where $\text{Diag}(K) := \{(x, x)\}_{x \in K} \subset K \times K$ **is the *diagonal of*** K**.**

Let $(y,z,\lambda) \in \mathrm{Capt}_{(4)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\mathrm{Diag}(K) \times \mathbb{R}_+))$ **belong to the capture basin. This means that there exist one forward evolution** $\overrightarrow{y}(\cdot) \in \overrightarrow{\mathcal{S}}^K(y)$ **viable in** K**, one backward evolution** $\overleftarrow{z}(\cdot) \in \overleftarrow{\mathcal{S}}^K(z)$ **viable in** K**, the evolution** $\lambda(t) := \lambda - \int_0^t \mathbf{l}(\overrightarrow{y}(t), \overrightarrow{u}(t))dt - \int_0^t \mathbf{l}(\overleftarrow{z}(t), \overleftarrow{v}(t))dt$ **and a time** $t^\star$ **such that,**

- **for all** $t \in [0,t^\star]$**,** $\overrightarrow{y}(t) \in K$**,** $\overleftarrow{z}(t) \in K$**,**

$$\lambda - \int_0^t \mathbf{l}(\overrightarrow{y}(t), \overrightarrow{u}(t))dt - \int_0^t \mathbf{l}(\overleftarrow{z}(t), \overleftarrow{v}(t))dt \;\geq\; 0$$

- **and** $\overrightarrow{y}(t^\star) = \overleftarrow{z}(t^\star)$ **and**

$$\lambda - \int_0^{t^\star} \mathbf{l}(\overrightarrow{y}(t), \overrightarrow{u}(t))dt - \int_0^{t^\star} \mathbf{l}(\overleftarrow{z}(t), \overleftarrow{v}(t))dt \;\geq\; \mathbf{c}(\overrightarrow{y}(t^\star), \overleftarrow{z}(t^\star))$$

Let us introduce the evolution $x(t)$ defined by $x(t) := \overrightarrow{y}(t)$ /. This evolution $x(\cdot)$ is continuous at $t^\star$ because $x(t^\star) = \overrightarrow{y}(t^\star) = \overleftarrow{z}(t^\star)$, belongs to $\mathcal{S}(y,z)$ since $x(0) = \overrightarrow{y}(0) = y$, $x(2t^\star) = \overleftarrow{z}(0) = z$ and is governed by the differential inclusion starting at y. Furthermore,

$$\left\{ \begin{array}{l} \lambda - \left(\int_0^{2t^\star} \mathbf{l}(x(t), u(t))dt \right) \\ = \ \lambda - \left(\int_0^{t^\star} \mathbf{l}(\overrightarrow{y}(t), \overrightarrow{u}(t))dt + \int_0^{t^\star} \mathbf{l}(\overleftarrow{z}(t), \overleftarrow{v}(t))dt \right) \\ \geq \ \mathbf{c}(x(t^\star), x(t^\star)) \end{array} \right.$$

This means that there exist $x(\cdot) \in \mathcal{S}(y,z)$ and $t^\star \geq 0$ such that

$$\mathbf{c}(x(t^\star), x(t^\star)) + \int_0^{2t^\star} \mathbf{l}(x(t), u(t))dt \ \leq \ \lambda$$

This implies in particular that

$$\left\{ \begin{array}{l} \mathbf{U_c}(y,z) \;:=\; \inf_{x(\cdot)\in\mathcal{S}^K(y,z),\, t^\star} \left(\mathbf{c}(x(t^\star),x(t^\star)) + \int_0^{2t^\star} \mathbf{l}(x(t),u(t))dt \right) \\ \leq \inf_{(y,z,\lambda)\in \mathrm{Capt}_{(4)}(\mathbb{R}_+\times K\times K, \mathcal{E}p(\mathbf{c})\cap(\mathbb{R}_+\times \mathrm{Diag}(K)))} \lambda \end{array} \right.$$

For proving the opposite inequality, we associate with any $\varepsilon > 0$ an evolution $x_\varepsilon(\cdot) \in \mathcal{S}^K(y,z)$, a control $u_\varepsilon(\cdot)$ and $t_\varepsilon^\star \geq 0$ such that

$$\left(\mathbf{c}(x_\varepsilon(t_\varepsilon^\star), x_\varepsilon(t_\varepsilon^\star)) + \int_0^{2t_\varepsilon^\star} \mathbf{l}(x_\varepsilon(t), u_\varepsilon(t))dt\right) \leq \mathbf{U_c}(y,z) + \varepsilon$$

and the function

$$\lambda_\varepsilon(t) := \mathbf{U_c}(y,z) + \varepsilon - \int_0^{2t} \mathbf{l}(x_\varepsilon(t), u_\varepsilon(t))dt$$

Introducing the forward parts $\overrightarrow{y}_\varepsilon(t) := x_\varepsilon(t)$ and $\overrightarrow{u}_\varepsilon(t) := u_\varepsilon(t)$ for $t \in [0, t_\varepsilon^\star]$ and backward parts $\overleftarrow{z}_\varepsilon(t) := x_\varepsilon(2t_\varepsilon^\star - t)$ and $\overleftarrow{v}_\varepsilon(t) := u_\varepsilon(2t_\varepsilon^\star - t)$, we observe that $(\overrightarrow{y}_\varepsilon(t), \overleftarrow{z}_\varepsilon(t), \lambda_\varepsilon(t))$ is a solution to the auxiliary system (4) starting at $(y, z, \mathbf{U_c}(y,z) + \varepsilon)$, viable in $K \times K \times \mathbb{R}_+$ and satisfying

$$\begin{cases} \lambda_{\varepsilon}(t) \ := \ \mathbf{U_c}(y,z) + \varepsilon - \displaystyle\int_0^{2t\star_{\varepsilon}} \mathbf{l}(x_{\varepsilon}(t), u_{\varepsilon}(t))dt \\ \geq \ \mathbf{c}(\overrightarrow{y}_{\varepsilon}(t\star_{\varepsilon}), \overleftarrow{z}_{\varepsilon}(t\star_{\varepsilon})) \end{cases}$$

This implies that $(y, z, \mathbf{U_c}(y, z) + \varepsilon)$ **belongs to the capture basin** $\mathrm{Capt}_{(4)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\mathrm{Diag}(K) \times \mathbb{R}_+))$**. Hence**

$$\inf_{(y,z,\lambda)\in \mathrm{Capt}_{(4)}(K\times K\times\mathbb{R}_+, \mathcal{E}p(\mathbf{c})\cap(\mathrm{Diag}(K)\times\mathbb{R}_+))} \lambda \ \leq \ \mathbf{U_c}(y,z) + \varepsilon$$

and it is enough to let ε **converge to** 0**.** ■

5 Collision Kernels

Eupalinian kernels are particular cases of *collision kernels* associated with a pair of evolutionary system $\mathcal{S}$ and $\mathcal{T}$ systems associated with control systems

$$\begin{cases} (i) & y'(t) = f(y(t), u(t)) \text{ where } u(t) \in U(y(t)) \\ (ii) & z'(t) = g(z(t), v(t)) \text{ where } v(t) \in V(z(t)) \end{cases} \tag{5}$$

Let $K \subset X$ and $L \subset X$ be two intersecting environments. We denote by $\mathcal{S}(y) \times \mathcal{T}(z)$ the set of evolutions $(y(\cdot), z(\cdot)) \in \mathcal{S}(y) \times \mathcal{T}(z)$ governed by the pair of evolutionary systems $\mathcal{S}$ and $\mathcal{T}$ viable in $K \times L$.

We say that they *collide* if there exists a finite *collision time* $t^\star \geq 0$ such that $y(t^\star) = z(t^\star) \in K \cap L$.

The *collision kernel* $\text{Coll}_{\mathcal{S},\mathcal{T}}(K, L) \subset K \times L$ is the subset of pairs $(y, z) \in K \times L$ such that there exist at least two viable colliding evolutions $(y(\cdot), z(\cdot) \in \mathcal{S}^K(y) \times \mathcal{T}^L(z)$.

If the collision time $t^\star$ is less than or equal to a prescribed time T, this subset is called T-*collision kernel* $\mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K,L)(T)$ and the set-valued map $T \rightsquigarrow \mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K,L)(T)$ the *collision tube.*

The collision kernel $\mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K,L)$ of $K \cap L$ under the evolutionary systems $\mathcal{S}$ and $\mathcal{T}$ associated with the systems (5), p.24 is the capture basin

$$\mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K,L) \ = \ \mathrm{Capt}_{(5)}(K \times L, \mathrm{Diag}(K \cap L))$$

of the diagonal of $K \cap L$ viable in $K \times L$ under the auxiliary system (5), p.24.

Take any $(y, z) \in \mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K, L)$**. The minimum collision time of viable colliding evolutions starting from** y **and** z **is the *collision function* defined by:**

$$\omega_{(K,L)}(y, z) \ := \ \inf \{T \text{ such that } (y, z) \in \mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K, L)(T)\}$$

Let $B \subset K$ **and** $C \subset L$ **be two sources, the function**

$$\omega_{(K,L)}(B, C) \ := \ \inf_{y \in B, \ z \in C} \omega_{(K,L)}(y, z)$$

is called the *collision distance* between the two sources B **and** C**, i.e., the minimal collision time of the viable colliding evolutions starting from** B **and** C**.**

Let us consider system

$$\left\{\begin{array}{l} \tau'(t) = -1 \\ y'(t) \; = \; f(y(t), u(t)) \text{ where } u(t) \; \in \; U(y(t)) \\ z'(t) \; = \; g(z(t), v(t)) \text{ where } v(t) \; \in \; V(z(t)) \end{array}\right. \tag{6}$$

The graph of the collision tube $\text{Coll}_{\mathcal{S},\mathcal{T}}(K,L)(\cdot)$ **is the viable-capture basin of** $\{0\} \times \text{Diag}(K \cap L)$ **viable in** $\mathbb{R}_+ \times K \times L$ **under the system (6):**

$$\text{Graph}(\text{Coll}_{\mathcal{S},\mathcal{T}}(K,L)(\cdot)) \; = \; \text{Capt}_{(6)}(\mathbb{R}_+ \times K \times L, \{0\} \times \text{Diag}(K \cap L))$$

or, equivalently,

$$\omega_{(K,L)}(y,z) \; = \; \inf_{(T,y,z) \in \text{Capt}_{(6)}(\mathbb{R}_+ \times K \times L, \{0\} \times \text{Diag}(K \cap L))} T$$

Consider a pair of evolutionary system $\mathcal{S}$ and $\mathcal{T}$ systems associated with control systems (5), p.24:

$$\begin{cases} (i) & y'(t) = f(y(t), u(t)) \text{ where } u(t) \in U(y(t)) \\ (ii) & z'(t) = g(z(t), v(t)) \text{ where } v(t) \in V(z(t)) \end{cases}$$

We look for *common solutions* $x(\cdot)$ of these two evolutionary systems (5). Whenever the control system (5)(i) is simpler to solve than the differential inclusion (5)(ii), the solutions of which are interpreted as *"particular" solutions*, *one can regard such common solutions to (5)(i) and (5)(ii) as particular solutions to the differential inclusion (5)(i).*

For instance,

- **taking $g(z,v) := 0$, the common solutions are equilibria of (5)(i),**
- **taking for $g(z,v) = v$ a constant velocity, then common solutions are affine functions of time t,**
- **taking for $g(z,v) = -mz$, then common solutions are exponential functions of time ze^{-mt}**

and so on. Finding particular solutions amounts to finding the set of the initial states from which common solutions do exist.

Then the set of points from which start common solutions to the control systems is the viability kernel $\mathrm{Viab}_{(5)}(\mathrm{Diag}(X))$ of the diagonal under (5).

6 Collision Optimization

We introduce a cost function $\mathbf{c} : X \times X \mapsto \mathbb{R} \cup \{+\infty\}$ (regarded as a *collision cost*) and a Lagrangian $\mathbf{l} : (y, z, u, v) \rightsquigarrow \mathbf{l}(y, z, u, v)$.

The optimal viable collision problem consists in finding colliding viable evolutions $y(\cdot) \in \mathcal{S}(y)$ and $z(\cdot) \in \mathcal{T}(z)$ and a time $t^\star \geq 0$ minimizing

$$\left\{ \begin{array}{l} \mathbf{W_c}(y, z) \quad = \quad \inf_{(y(\cdot), z(\cdot)) \in \mathrm{Coll}(y,z),\, t^\star \mid y(t^\star) = z(t^\star)} \\ \left(\mathbf{c}(y(t^\star), z(t^\star)) + \int_0^{t^\star} \mathbf{l}(y(t), z(t), u(t), v(t)) dt \right) \end{array} \right.$$

By taking $\mathbf{c} \equiv 0$ and $\mathbf{l}(y, z, u, v) \equiv 1$, we find the problem of governing two evolutions in minimal time.

Let us consider the auxiliary control system

$$\left\{\begin{array}{lll} (i) & y'(t) = f(y(t),u(t)) \text{ where } u(t) \in U(y(t)) \\ (ii) & z'(t) = g(z(t),v(t)) \text{ where } v(t) \in V(z(t)) \\ (iii) & \lambda'(t) = -\mathbf{l}(y(t),z(t),u(t),v(t)) \end{array}\right. \quad (7)$$

Then

$$\mathbf{W_c}(y,z) = \inf_{(y,z,\lambda)\in \mathrm{Capt}_{(7)}(K\times K\times\mathbb{R}_+, \mathcal{E}p(\mathbf{c})\cap(\mathrm{Diag}(K)\times\mathbb{R}_+))} \lambda$$

where $\mathrm{Diag}(K) := \{(x,x)\}_{x\in K} \subset K\times K$ **is the** ***diagonal of*** K**.**

Let $(y, z, \lambda) \in \text{Capt}_{(7)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\text{Diag}(K) \times \mathbb{R}_+))$ **belong to the capture basin. This means that there exist one evolution** $y(\cdot) \in \mathcal{S}^K(y)$ **viable in** K**, one evolution** $z(\cdot) \in \mathcal{T}^L(z)$ **viable in** L**, the evolution** $\lambda(t) := \lambda - \int_0^t \mathbf{l}(y(s), z(s), u(s), v(s))ds$ **and a time** $t^\star$ **such that,**

- **for all** $t \in [0, t^\star]$**,** $y(t) \in K$**,** $z(t) \in K$**,**

$$\lambda - \int_0^t \mathbf{l}(y(s), z(s), u(s), v(s))ds \ \geq \ 0$$

- **and** $y(t^\star) = z(t^\star)$ **and**

$$\lambda - \int_0^{t^\star} \mathbf{l}(y(s), z(s), u(s), v(s))ds \ \geq \ \mathbf{c}(y(t^\star), z(t^\star))$$

This implies that

$$\mathbf{W_c}(y,z) \;\leq\; \mathbf{c}(y(t^\star),z(t^\star)) + \int_0^{t^\star} \mathbf{l}(y(s),z(s),u(s),v(s))ds \;\leq\; \lambda$$

and thus, that

$$\mathbf{W_c}(y,z) \;\leq\; \inf_{(y,z,\lambda)\in \mathrm{Capt}_{(7)}(K\times K\times\mathbb{R}_+,\mathcal{E}p(\mathbf{c})\cap(\mathrm{Diag}(K)\times\mathbb{R}_+))} \lambda$$

For proving the opposite inequality, we associate with any $\varepsilon > 0$ **two colliding evolutions** $y_\varepsilon(\cdot) \in \mathcal{S}^K(y)$ **and** $z_\varepsilon(\cdot) \in \mathcal{T}^L(y)$ **at some time** $t_\varepsilon^\star \geq 0$**, controls** $u_\varepsilon(\cdot)$ **and** $v_\varepsilon(\cdot)$ **such that**

$$\left(\mathbf{c}(y_\varepsilon(t_\varepsilon^\star),z_\varepsilon(t_\varepsilon^\star)) + \int_0^{t_\varepsilon^\star} \mathbf{l}(y_\varepsilon(t),z_\varepsilon(t),u_\varepsilon(t),v_\varepsilon(t))dt\right) \;\leq\; \mathbf{W_c}(y,z)+\varepsilon$$

and the function

$$\lambda_\varepsilon(t) \;:=\; \mathbf{W_c}(y,z)+\varepsilon - \int_0^t \mathbf{l}(y_\varepsilon(s),z_\varepsilon(s),u_\varepsilon(s),v_\varepsilon(s))ds$$

By construction,

$$\lambda_\varepsilon(t_\varepsilon^\star) \;\geq\; \mathbf{c}(y(t_\varepsilon^\star), z(t_\varepsilon^\star)) \text{ and } y(t_\varepsilon^\star) \;=\; z(t_\varepsilon^\star)$$

This implies that $(y, z, \mathbf{W_c}(y, z) + \varepsilon)$ **belongs to the capture basin** $\mathrm{Capt}_{(7)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\mathrm{Diag}(K) \times \mathbb{R}_+))$**. Hence**

$$\inf_{(y,z,\lambda)\in \mathrm{Capt}_{(7)}(K\times K\times\mathbb{R}_+,\mathcal{E}p(\mathbf{c})\cap(\mathrm{Diag}(K)\times\mathbb{R}_+))} \lambda \;\leq\; \mathbf{W_c}(y, z) + \varepsilon$$

and it is enough to let ε **converge to** 0**.** ■

Merci pour votre attention
Thanks for your attention

Perfectly preserved masonry | Stone reinforcement | The "landslide" | Wet section with stalactites

Fig.11: Through the Eupalinos tunnel: Further down in the north section.