# Indices de Complexité Connexionnistes : Simplicité connexionniste sous-jacente à des modèles dynamiques de décentralisation par les prix en économie

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**although there is no consensus on the definition of complexity. However, reading the literature on complexity, and quoting George Cowan, the founder of the Santa Fe Institute, "in the universe, everything is connected with every thing" seems to be the consensual agreement of the members of this Institute.**

**This attempt to sustain the viability of the system by connecting the dynamics or the constraints of its agents may be a general feature of "complex systems".**

**However, Seth Loyd had found 31 different definitions of complexity at the beginning of the 90's. This number increased a lot since. Complexity is indeed a polysemous word, that tries to embrace too many distinct phenomenon of interest in biological and social sciences.**

**Physicists have attempted to measure "complexity" in various ways, through the concept of Clausius's entropy, Shannon's information, the degree of regularity instead of randomness, "hierarchical complexity" in the display of level of interactions, "grammatical complexity" measuring the language to describe it, temporal or spatial computational, measuring the computer time or the amount of computer memory needed to describe a system, etc.**

**The problem is that living systems being open, there is no way to describe them entirely and consequently, to demonstrate the truth of any proposition. We are left with metaphors, and among them, mathematical ones, to validate a consensual definition of one meaning — actually, one submeaning, so to speak — of complexity.**

1. **Connectionist Complexity and ... Price Decentralization**
2. **Measure of Emergence of Complexity**
3. **Hierarchical Complexity**
4. **Links with Networks**
5. **Parisi's Complexity in the Framework of Exploration of Energy Landscapes (with Annick Lesnes)**

**John Horgan (from Complexity to Perplexity, The End of Science), has criticized a combination of the four Cs, which consist of cybernetics, catastrophe theory, chaos theory, and complexity theory.**

**They are fads, intellectual bubbles of little consequence. They would soon disappear and deservedly so, once scholars and intellects realized what worthless dross they truly are. The four Cs are linked through their common use of nonlinear dynamical systems.**

# 1 Connectionist Complexity

**We shall investigate one aspect of complexity closely related to the decentralization issue, connectionist complexity, as one answer to adapt to more and more viability constraints, which implies the emergence of links between the agents of the dynamical economy and their evolution.**

**We compare it with other dynamical decentralization mechanisms, such as price decentralization.**

**The emergence of the cyberspace may give a new life to connectionist mechanisms, since any two agents would soon be able to communicate at each instant without the mediation of a price or other messages providing the necessary information.**

**Assume for instance that, if there were no scarcity constraints to comply with, the dynamical behavior of each consumer $i = 1, \ldots, n$ would be fully decentralized, and the evolution of its consumption $x_i(t) \in Y$ at time $t \geq 0$ would be governed by a differential equation of the form**

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; g_i(x_i(t))$$

**There is no reason why the scarcity constraints**

$$\forall\, t \geq 0, \;\; \sum_{i=1}^{n} x_i(t) \; \in \; M$$

**of a collective nature would be satisfied by such independent behaviors.**

**One first solution which comes up to the mind is to "connect" these dynamics through a connection matrix $W := (w_i^j) \in \mathcal{L}(X, X)$ and to govern the evolution of consumptions according to an evolution law of the form**

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; \sum_{j=1}^{n} w_j^i(t) g_j(x_j(t))$$

**where $w_i^j \in \mathcal{L}(Y, Y)$. Naturally, we obtain a fully decentralized economy when the connection matrix is the identity matrix.**

**We propose mathematical metaphors designing evolution laws governing the evolution of connection matrices linking the autonomous dynamics of each agent in order to sustain the viability constraints prescribed by the environment of the isolated system.**

**Our point is the following: Starting with a disconnected system — for which the nondiagonal elements of the connection matrix are all equal to zero — the viability can be maintained whenever each agent of the system is linked with other through an evolving connectionist matrix, dictating cooperation among the agents.**

**The larger the number of independent constraints, the more numerous the links between any agent with the others. In the mathematical version of the metaphor, a class of evolution laws governing the evolution of these connection matrices can be designed from the disconnected dynamical system and the set of constraints.**

**We regard here *connectionism* — a less normative and more neutral term than *cooperation* whenever the system arises in economics or biology — as an answer to adapt to more and more viability constraints, which implies the emergence of links between the components of a system and their evolution. A system is** disconnected (or autonomous, free, autarchic, decentralized, etc.) **if the connection matrix is the identity (or unit) matrix. In some loose words, the distance between the** connection matrix **— which is the matrix linking each component of the system to the others — and the identity matrix should capture and measure the concept of an** index of (connectionist) complexity**. The larger such a connectionist complexity index, the more complex — or labyrinthine or intricate — the connectionist feature of the system.**

**The norm of $\|W'(t)\|$ of the velocity $W'(t)$ of a time dependent connection matrix $W(t)$ starting from the identity matrix provides a measure of the emergence of connectionist at time $t$.**

**One can also measure other features of connectionist complexity through the sparsity of the connection matrix, i.e., the number — or the position — of entries which are equal to zero or "small". The sparser such a connectionist matrix, the less complex the system.**

**We shall single out for instance the problem of** maximal decentralization **which requires to find connecting matrices $W(t)$ as close as possible to the identity matrix $1$**

**— slow evolution —**

**or which evolve as slowly as possible while sustaining the viability of the allocation of scarce resources**

**— heavy evolution —**

**in such a way that the connection matrix remains constant as long as the viability of the system is not at stakes.**

**We advocate the latter mode, which has the property of locking in the connection matrices which can regulate forever a nonempty subset of states, called the viability niche of the connection matrix.**

**Therefore, scientific activity begins by dividing a system into two classes, the system under study and its environment. This division is always arbitrary, but often justified by the scientists in quest of explanation. Once conceptually isolated from its environment, a living system fuels itself in the last analysis on solar energy through the consumptions of wastes of the other components of the open system set apart in the description of the environment. Each component of the system which can evolve independently in the absence of constraints, must interact each other in order to maintain the viability of the system imposed by its environment.**

**This attempt to sustain the viability of the system by connecting the dynamics of its agents may be a general feature of "complex systems".**

**Is not complexity meaning in the day-to-day language the labyrinth of connections between the components of a living organism or organization or system ? Is not the purpose of complexity to sustain the constraints set by the environment and its growth parallel to the increase of the web of constraints ?**

**Economic history has shown an everlasting trend toward a highly connected network of labyrinthine — connectionist — complexity. In this sense, complexity arose with the apparition of life and seems necessary to the pursuit of its evolution at all levels of organization.**

**Compounded with this issue of connectionist complexity — that is naturally involved in neural networks — is the question of whether these links relating one agent to another can be subsumed by regulation parameters — which we call in short regulons — and whether and when they provide the same evolutions of the state of the system. Such regulons are messages sent to the consumers, and the simplest ones are the prices.**

In a decentralized mechanism — actually, we should say in an "a-centralized mechanism", since we do not assume the existence of a "center" — the information on the alloaction problem is split and mediated by, say, a "message" which summarizes part of the information. In our case, we use the "price" $p$ as a main example of message (actually, the message with the smallest dimension). Knowing the price $p$, consumers are supposed to know how to choose their consumption bundle, without

- knowing the behavior of their fellow consumers
- knowing the set of scarce resources

**There are many other decentralized models, such as "rationing" mechanisms which involve shortages (and lines), or "frustration" of consumers, or "monetary" mechanisms, or others.**

**Naturally, there is no "pure" decentralization, since the choice of the decentralization message is in some sense centralized. The prices help consumer to make their choice in a decentralized way, but the difficulty is postponed to explain the evolution of price.**

**In order to comply to viability constraints, we investigate successively**

1. **the regulation by subtracting prices to the original dynamical behavior**

$$x'(t) \;=\; g(x(t)) - p(t)$$

2. **the regulation by connecting the agents of the original dynamics through connecting matrices**

$$x'(t) \;=\; W(t)g(x(t))$$

3. **a combination of both regulation procedures**

$$x'(t) \;=\; W(t)g(x(t)) - p(t)$$

**We also need to "differentiate" viability constraints, and for that purpose, we need to implement the concept of tangency to any subset.**

**The contingent cone $T_M(x)$ to $M \subset Y$ at $y \in M$ is the set of directions $v \in Y$ such that there exist sequences $h_n > 0$ converging to $0$ and $v_n$ converging to $v$ satisfying $y + h_n v_n \in M$ for every $n$. The normal cone is**

$$N_M(y) := \{q \in Y^* \mid \forall\, v \in T_M(y),\ \langle q, v \rangle \leq 0\}$$

**For simplicity, we shall assume that $M$ is convex.**

**We recall that a function** $x(\cdot) : I \mapsto X$ **is said to be** viable in $K$ **if and only if**

$$\forall\, t \geq 0, \ \ x(t) \ \in \ K$$

**Then a differentiable viable function in** $K$ **satisfies**

$$\forall\, t \geq 0, \ \ x'(t) \ \in \ T_K(x(t))$$

**If** $K := h^{-1}(M)$ **where** $h : X \mapsto Y$ **is a continuously differentiable map such that** $h'(x)$ **is surjective and** $M$ **is closed and convex (or, more generally, sleek), then**

$$T_K(x) \ = \ h'(x)^{-1} T_M(h(x))$$

**As the simplest example, we look for ways of coordinating the decisions of the consumers by subtracting prices to their dynamical priceless behavior**

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; g_i(x_i(t)) - p_i(t)$$

**in order that viability constraints of the form**

$$\forall\, t \geq 0, \;\; h(x_1(t), \ldots, x_n(t)) \; \in \; M$$

**are always satisfied.**

**We define the Regulation Map** $\Pi_M$ **by**

$$\Pi_M(x) \ := \ \{p \mid h'(x)(g(x)-p) \in T_M(h(x))\}$$

**The prices** $p(t) := (p_1(t), \ldots, p_n(t))$ **regulating viable solutions are given by the regulation law**

$$p(t) \ \in \ \Pi_M(x_1(t), \ldots, x_n(t))$$

**In the case of the problem of allocations of scarce resources**

$$\forall\, t \geq 0, \quad \sum_{i=1}^{n} x_i(t) \;\in\; M$$

**for instance, we saw that**

$$\Pi_M(x) \;:=\; \frac{1}{n}\left(T_M\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} g_i(x_i)\right)$$

**For building feedback prices, we can for instance think of explicitly selecting some prices of the regulation map, for instance, the price $\varpi^\circ(x) \in \Pi_M(x)$ with minimal norm. Viable solutions obtained with this feedback price are called** slow viable solutions.

**When** $B \in \mathcal{L}(X, Y)$ **is surjective, its orthogonal right inverse is equal to**

$$B^{+} \;=\; B^{\star}(BB^{\star})^{-1}$$

**that we can supply** $Y$ **with the final norm** $\mu^{B}$ **defined by** $\mu^{B}(z) := \|B^{+}z\|$ **and that we denote by** $\pi_{K}^{B}$ **the projector of best approximation onto the closed subset** $K$ **for this final norm.**

**In this case, the unique solution** $\bar{x}$ **to the minimization problem**

$$\inf_{Bx \in K+v} \|x - u\|$$

**is equal to**

$$\bar{x} \;=\; u - B^{+}(1 - \pi_{M}^{B})(Bu - v)$$

**When $M$ is convex (or more generally, sleek), then its tangent cones $T_M(y)$ are convex.**

**In this case, since the polar cone to the contingent cone $T_M(y)$ is the normal cone $N_M(y)$, we also know that we can write the solution to the minimization problem**

$$\inf_{Bx \in T_M(y)+v} \|x-u\|$$

**is equal to**

$$\bar{x} \;=\; u - B^{+}(1-\pi^{B}_{T_M(y)})(Bu-v)$$

**It is also equal to**

$$\bar{x} \;=\; u - B^{\star}\pi^{B^{\star}}_{N_M(y)}(BB^{\star})^{-1}(Bu-v)$$

**where $\pi^{B^{\star}}_{N_M(y)}$ denotes the projector onto the normal cone $N_M(y)$ when the dual $Y^{\star}$ is supplied with the "dual final scalar product".**

**Let us assume that**

$$\forall\, x \in K,\quad h'(x) \text{ is surjective} \tag{1}$$

**and that $M$ is closed convex (or more generally, sleek). Then the slow solution of the dynamical economy**

$$x'(t) \;=\; g(x(t)) - p(t) \tag{2}$$

**subjected to the viability constraints**

$$\forall\, t \geq 0,\quad h(x(t)) \;\in\; M$$

**is the solution to the differential equation**

$$x'(t) \;=\; g(x(t)) - \varpi^{\circ}(x(t))$$

**where**

$$\left\{ \begin{array}{l} \varpi^{\circ}(x) \\ =\; h'(x)^{+}\left(1 - \pi_{T_M(h(x))}^{h'(x)}\right) h'(x)g(x) \\ =\; h'(x)^{\star}\pi_{N_M(h(x))}^{h'(x)^{\star}}(h'(x)h'(x)^{\star})^{-1}h'(x)g(x) \end{array} \right. \tag{3}$$

**We look for the problem of sustaining viability of the evolution by connecting each consumer with the other ones through a connection matrix** $W := (w_i^j) \in \mathcal{L}(X, X)$**:**

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; \sum_{j=1}^{n} w_i^j(t) g_j(x_j(t))$$

**where** $w_i^j(t) \in \mathcal{L}(Y, Y)$**. We write it in the form**

$$x'(t) \; = \; W(t) g(x(t)) \tag{4}$$

**Naturally, we obtain a fully decentralized economy when the connection matrix is the identity matrix. But this is not always the case, and consumers may take into account the behavior of the other consumers in their dynamics. Then the new regulation parameter is no longer a price $p \in X^\star$, but a** connection matrix $W$**.**

**We introduce the new regulation map**

$$R_M(x) \; := \; \{W \in \mathcal{L}(X,X)$$

**such that**

$$h'(x)Wg(x) \; \in \; T_M(h(x))\}$$

**We can reestablish viability by involving viability multipliers $q \in Y^*$ ranging over the dual $Y^* := Y$ of the resource space $Y$ (identified with $Y$ for simplicity) .**

**We denote by $W^* \in \mathcal{L}(Y^*, X^*)$ the transpose of $W$:**

$$\forall\, q \in Y^*,\ \forall\, x \in X,\ \ \langle W^* q, x\rangle \ := \ \langle q, Wx\rangle$$

**We denote by $x \otimes q \in \mathcal{L}(X^\star, Y^*)$ denotes the tensor product defined by**

$$x \otimes q \ :\ p \in X^* := X \mapsto (x \otimes q)(p) \ := \ \langle p, x\rangle q$$

**the matrix of which is made of entries**

$$(x \otimes q)_i^j = x_i q^j$$

**A connection matrix of the form**

$$W(x) := 1 - \frac{g(x)}{\|g(x)\|^2} \otimes \varpi(x)$$

**the entries of which are equal to**

$$w_{ij}(x) \ = \ \delta_{i,j} - \frac{g(x)_i \varpi(x)_j}{\|g(x)\|^2}$$

**belongs to** $R_M(x)$ **if and only if** $\varpi(x)$ **belongs to** $\Pi_M(x)$ **and the viable solutions to the differential equations** $x' = W(x)g(x)$ **and** $x' = g(x) - \varpi(x)$ **do coincide.**

**Indeed, we observe that**

$$W(x)g(x) \;=\; g(x) - \frac{\langle g(x), g(x)\rangle}{\|g(x)\|^2}\varpi(x) \;=\; g(x) - \varpi(x)$$

**so that the two differential equations $x' = W(x)g(x)$ and $x' = g(x) - \varpi(x)$ are the same. Furthermore, to say that $W(x) \in R_M(x)$ amounts to saying that**

$$h'(x)W(x)g(x) \;=\; h'(x)g(x) - h'(x)\varpi(x) \;\in\; T_M(h(x))$$

**so that the viability conditions are the same.** $\square$

**In other words, the regulation by connection matrices of the form**

$$W(x) := 1 - \frac{g(x)}{\|g(x)\|^2} \otimes \varpi(x)$$

**is equivalent to the price decentralization mechanism.**

**Let us assume that**

$$\forall\, x \in K, \quad h'(x) \text{ is surjective} \tag{5}$$

**and that $M$ is closed convex (or sleek). The solution with maximal decentralization is governed by the differential equation**

$$x'(t) \;=\; W^{\circ}(x(t))g(x(t))$$

**where $W^{\circ}(x)$**

$$1 - \frac{g(x)}{\|g(x)\|^2} \otimes h'(x)^{\star}\pi^{h'(x)^{\star}}_{N_M(h(x))}(h'(x)h'(x)^{\star})^{-1}(h'(x)g(x)) \tag{6}$$

**Furthermore, the slow viable solutions to dynamical economy regulated by prices and the viable solutions to the connection economy under maximal decentralization coincide.**

## 2 Inertia Function: A Measure of Emergence

**Consider a system of the form**

$$\begin{cases} (i) & x'(t) \in W(t)g(x(t)) \\ (ii) & \text{where } x(t) \in K(W(t)) \end{cases} \tag{7}$$

**Denote by** $\mathcal{P}(x, W)$ **the set of evolutions** $(x(\cdot), W(\cdot))$ **governed by (7) starting at** $(x, W)$**.**

**The inertia function** $\alpha$ **is defined by**

$$\alpha(T, x, W) = \inf_{(x(\cdot), W(\cdot)) \in \mathcal{P}(x, W)} \sup_{t \in [0, T]} \|W'(t)\| \tag{8}$$

**describing the smallest of the worst measure of emergence of complexity on the time interval** $[0, T]$**.**

**The epigraph of the inertia function is the capture basin of the target** $\{0\} \times \mathrm{Graph}(K^{-1} \times \mathbb{R}_+)$ **viable in** $\mathbb{R}_+ \times \mathrm{Graph}(K^{-1} \times \mathbb{R}_+)$ **under the metasystem**

$$\left\{ \begin{array}{lrcl} (i) & \tau'(t) & = & -1 \\ (ii) & x'(t) & = & W(t)g(x(t)) \\ (ii) & W'(t) & = & U(t) \\ (ii) & y'(t) & = & 0 \\ & \text{where } \|U(t)\| & \leq & y(t) \end{array} \right. \tag{9}$$

**Then, for every given emergence threshold** $c$**, the evolution of connection matrices is then governed by the adjustment law**

$$W'(t) \in G(T-t, x(t), W(t), c)$$

**where** $G(t,x,W,c)$ **is the set of connection matrices** $U$ **such that**

$$(-1, Wg(x), U, 0) \in T_{\mathcal{E}p(\alpha)}(t,x,W,c)$$

**The inertia function is lower semicontinuous and a generalized solution (in the Frankowska-Barron-Jensen sense) to Hamilton-Jacobi-Bellman partial differential equation**

$$-\frac{\partial \alpha(t,x,W)}{\partial t} + \sum_{i,j=1}^{n} \frac{\partial \alpha(t,x,W)}{\partial x_i} w_i^j g_j(x_j) - \alpha(t,x,W) \sum_{i,j=1}^{n} \left| \frac{\partial \alpha(t,x,W)}{\partial w_i^j} \right| \; = \; 0$$

**on** $\mathbb{R}_+ \times \mathrm{Graph}(K^{-1})$**.**

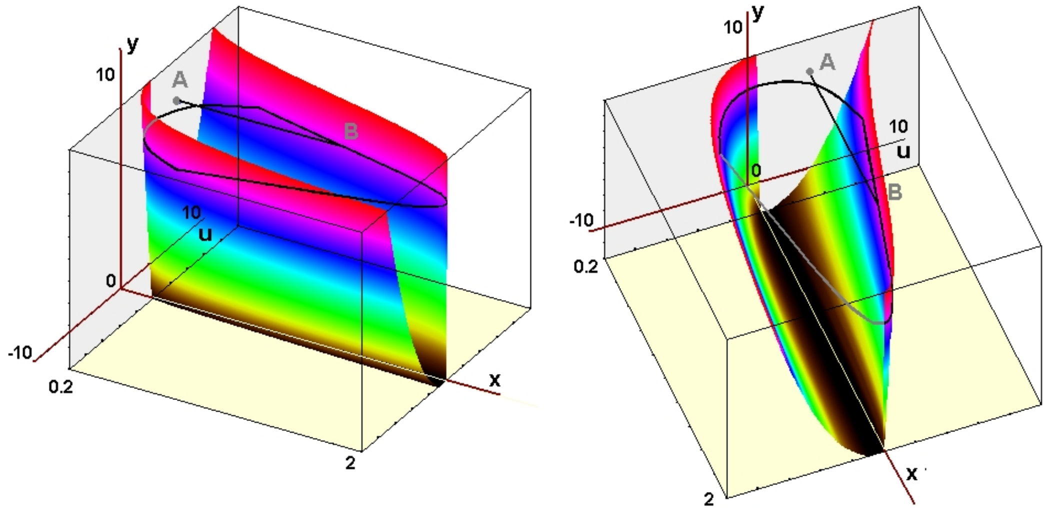



Figure 1: **Fonction d'inertie**

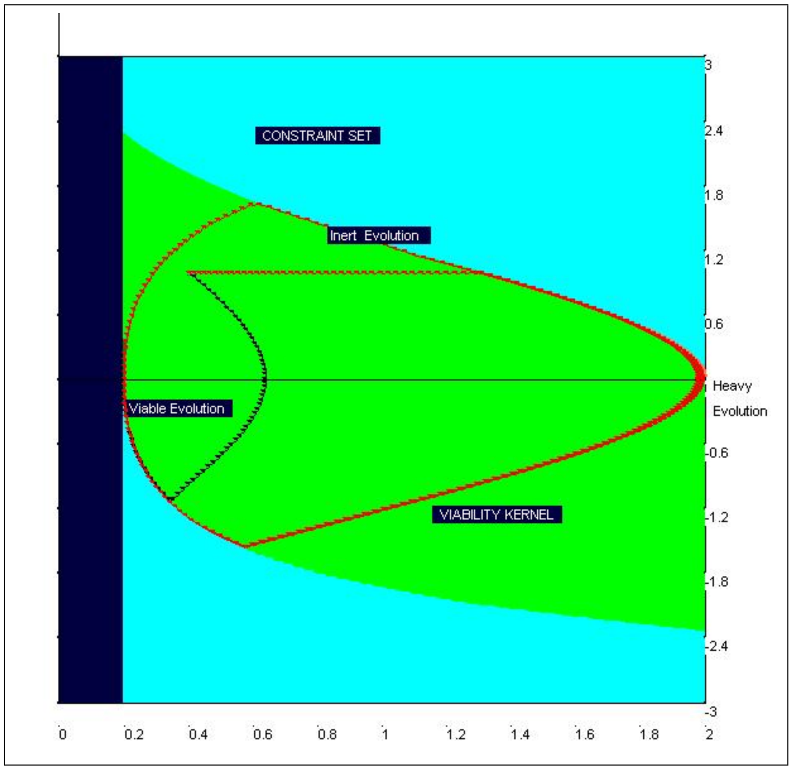
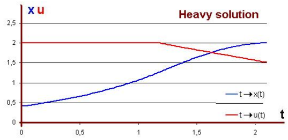



Figure 2: **Coupe de la fonction d'inertie** .

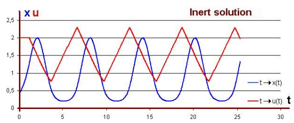
Figure 3:




**Décours temporel de l'évolution lourde**

Cette figure indique les courbes de l'évolution lourde (en bleu) et du taux (en rouge) déduites de l'algorithme de viabilité en même temps que la fonction d'inertie. Le taux demeure constant jusqu'à ce que la trajectoire de la solution exponentielle heurte la frontière nord (Figure 2) et ralentit ensuite à vitesse constante jusqu'à ce que l'état atteigne la borne supérieure $b$. Source: Patrick Saint-Pierre.

Figure 4:




**Décours temporel de l'évolution hystérétique**

Cette figure indique les courbes de l'évolution hystérétique (en bleu) et du taux (en rouge) déduites de l'algorithme de viabilité en même temps que la fonction de flexibilité. Les vitesses des taux sont alternativement positives et négatives, égales au seuil de flexibilité (Figure 2). Source: Patrick Saint-Pierre.

# 3 Hierarchical Complexity

**We next briefly investigate the intricate case when the connection matrices are involved in the constraints in a hierarchical process of the form**

$$W_{m-1} \cdots W_j \ldots W_0 x - x_m \in M$$

**This describes both chemical or biological "cycles" where the** $m$ **organisms are related by constraints** $x_j = W_j x_{j-1}$ **and** $x_m \in x_{m-1} - M$

**Instead of imposing a given sequence of connection matrices and studying the evolution of the commodities under dynamical systems of the form**

$$x_j'(t) = g_j(x_j(t))$$

**we want to derive also the evolution of connection matrices** $W_j(t)$ **in such a way that the above viability constraint is satisfied at each instant. This involved situation is best understood if it is illustrated by some examples.**

**One aspect of the biological and ecological problems of "co-evolution" and "co-viability" can be translated (in its simplest form) in the following way. The states $x_0(\cdot)$ and $x_1(\cdot)$ of two organisms evolve in state spaces $X_0$ and $X_1$. For that purpose, the organism # 1 consumes $\langle C_1, x_1 \rangle$ units of (scalar) "metabolites" (where $C_1 \in X_1^\star$) which are provided by the activity $\langle A_0, x_0 \rangle$ of organism # 0 proportionally to its state $x_0$ (where $A_0 \in X_0^\star$). So, for surviving, organism # 1 needs the activity of $w$ units of organisms # 0 for satisfying the "co-viability" constraint**

$$w_0 \langle A_0, x_0 \rangle - \langle C_1, x_1 \rangle \ \leq \ 0$$

**When the "metabolites" range over a vector space $Y_1$ instead of R, then the constraint becomes**

$$W_0 A_0 x_0 - C_1 x_1 \ \leq \ 0$$

**where $A_0$ and $C_1$ are given linear operators and where the connection matrix $W_0$ enters the co-viability constraints. Hence, we have to devise co-evolutions $x_0(\cdot)$ and $x_1(\cdot)$ of both the organisms as well as the evolution $W_0(\cdot)$ of the connection matrix such that the co-viability constraint is satisfied at each instant.** $\square$

**We regard the finite dimensional vector spaces $X$ and $Y$ as the input (resource) and the output space respectively and the space $\mathcal{L}(X,Y)$ of linear operators $W$ as input-output (production, connectionist) operators**

**Let $M \subset Y$ be the "demand set" that the producer must fulfill.**

**The problem takes the form:**

$$\forall\, t \geq 0, \;\; W(t)x(t) \;\in\; M$$

**where both the resource $x$ and the connectionist operator $W$ evolve. We start with a dynamic system of the form**

$$\begin{cases} i) \quad x'(t) \;=\; c(x(t)) \\ ii) \;\; W'(t) \;=\; \alpha(W(t)) \end{cases}$$

**under which the above constraints are not necessarily viable.**

**The viability of the constraints**

$$\forall\, t \geq 0, \;\; W(t)x(t) \; \in \; M$$

**can be reestablished when the initial system is replaced the above system by the control system**

$$\left\{ \begin{array}{lll} (i) & x'(t) & = \; c(x(t)) - W^{*}(t)q(t) \\ (ii) & W'(t) & = \; \alpha(W(t)) - x(t) \otimes q(t) \end{array} \right.$$

**which involves viability multipliers $q(t) \in Y^{\star}$ which satisfy the regulation law**

$$\forall\, t \geq 0, \;\; q(t) \; \in \; R_M(W(t), x(t))$$

**where the regulation map $R_M$ is defined by**

$$R_M(W, x) \; = \; (WW^{\star} + \|x\|^2 \mathsf{I})^{-1}(Wc(x) + \alpha(W)(x) - T_M(W(x)))$$

**One can even require that on top of it, the viability multiplier satisfies**

$$q(t) \; \in \; N_M(W(t)x(t)) \cap R_M(W(t), x(t)))$$

**The norm $\|q(t)\|$ of the viability multiplier $q(t)$ measures the intensity of the viability discrepancy of the dynamic since**

$$\begin{cases} (i) & \|c(x(t)) - x'(t)\| \ \leq \ \|W^*(t)\| \, \|q(t)\| \\ (ii) & \|\alpha(W(t)) - W'(t)\| \ = \ \|x(t)\| \, \|q(t)\| \end{cases}$$

**The viability multipliers of minimal norm in the regulation map provide the smallest velocities of the corrections, and thus, heavy evolutions.**

**Differential equation**

$$W'(t) \ = \ \alpha(W(t)) - x(t) \otimes q(t)$$

**involves the** Hebbian rule **proposed by Hebb in his** The organization of behavior **in 1949 as the basic learning process of synaptic weight:**

**Taking** $\alpha(W) = 0$**, the evolution of the** synaptic matrix $W := (a_i^j)$ **obeys the differential equation**

$$\frac{d}{dt} a_i^j(t) \ = \ -x_i(t) q^j(t)$$

The velocity of the synaptic weight is the product of the presynaptic activity and the post-synaptic activity.

**Observe that in this case**

$$\|W'(t)\| \ = \ \|x(t)\| \|q(t)\|$$

**The inertia of the connection matrix, which can be regarded as a index of dynamic connectionist complexity, is proportional to the norm of the viability multiplier.**

**The constraints are of the form**

$$W_{\mathbb{H}}^{\mathbb{H}-1} \cdots W_h^{h-1} \dots W_2^1 x_1 \ \in \ M_{\mathbb{H}}$$

**This describes for instance a production process associating with the resource** $x_1$ **the intermediate outputs** $x_2 := W_2^1 x_1$**, which itself produces an output** $x_3 := W_2^1 x_2$**, and so on, until the final output** $x_{\mathbb{H}} := W_{\mathbb{H}}^{\mathbb{H}-1} \cdots W_h^{h-1} \dots W_2^1 x_1$ **which must belong to the production set** $M_{\mathbb{H}}$ **.**

**The evolution without constraints of the commodities and the operators is governed by dynamical systems of the form**

$$\begin{cases} (i) \quad x_h'(t) \ = \ c_h(x_h(t)) \\ \\ (ii) \quad \dfrac{d}{dt} W_{h+1}^h(t) \ = \ \alpha_{h+1}^h(W_h(t)) \end{cases}$$

**The constraints**

$$\forall\, t \geq 0,\;\; W_{\mathbb{H}}^{\mathbb{H}-1}(t)\cdots W_{h}^{h-1}(t)\ldots W_{2}^{1}(t)x_1(t) \;\in\; M_{\mathbb{H}}$$

**are viable under the system**

$$\begin{cases} x_1'(t) \;=\; c_1(x_1(t)) + W_2^1(t)^\star(t)p^1(t) & (h=1) \\ x_h'(t) \;=\; c_h(x_h(t)) - p^{h-1}(t) + W_{h+1}^h(t)^\star p^h(t) & (h=1,\ldots,\mathbb{H}-1) \\ x_{\mathbb{H}}'(t) \;=\; c_{\mathbb{H}}(x_{\mathbb{H}}(t)) - p^{\mathbb{H}-1}(t) & (h=\mathbb{H}) \\ \dfrac{d}{dt}W_{h+1}^h(t) \;=\; \alpha_{h+1}^h(W_h(t)) + x_h(t)\otimes p^h(t) & (h=1,\ldots,\mathbb{H}-1) \end{cases}$$

**involving viability multipliers $p^h(t)$ (intermediate "shadow price"). The input-output matrices $W_{h+1}^h(t)$ obey dynamics involving the tensor product of $x_h(t)$ and $p^h(t)$.**

**We assume that** $X := \prod_{h=1}^{\mathbb{H}}$ **and** $Y := \prod_{k=1}^{\mathbb{K}}$ **and that** $W := (W_h^k)$ **where** $W_h^k \in \mathcal{L}(X_k, Y_h)$**. We introduce a set-valued map** $J : \{1, \ldots, \mathbb{H}\} \rightsquigarrow \{1, \ldots, \mathbb{K}\}$**.**

**The constraints are defined by**

$$\forall\ h = 1, \ldots, \mathbb{H},\ \ \sum_{k \in J(h)} W_h^k(t) x_k(t)\ \in\ M_h\ \subset\ Y_h$$

**We consider a system of differential equations**

$$\begin{cases} (i) \quad x_h'(t)\ =\ c_h(x_h(t)),\ \ h = 1, \ldots, \mathbb{H} \\ \\ (ii)\ \ \dfrac{d}{dt} W_h^k(t)\ =\ \alpha_h^k(W_h^k(t)) \end{cases}$$

**Then the constraints**

$$\forall\ h = 1, \ldots, \mathbb{H}, \ldots \sum_{k \in J(h)} W_h^k(t) x_k(t) \ \in\ M_h \ \subset\ Y_h$$

**are viable under the corrected system**

$$\begin{cases} (i) \quad x'_h(t) \ = \ c_h(x_h(t)) - \sum_{k \in J^{-1}(h)} W_k^h(t)^\star p^k, \quad h = 1, \ldots, \mathbb{H},\ k = 1, \ldots, \mathbb{K} \\ (ii) \ \ \dfrac{d}{dt} W_h^k(t) \ = \ \alpha_h^k(W_h^k(t)) - x_k(t) \otimes p^h(t),\ (h, k) \in \mathrm{Graph}(J) \end{cases}$$

**Let us consider the case when the set of allocations**

$$K \ := \ \left\{ x \in \prod_{i=1}^{n} L_i \mid \sum_{i=1}^{n} x_i \in M \right\}$$

**of scarce resources** $y \in M$ **is empty. We regard here** $M \subset Y := \mathbf{R}^n$ **as the production set of one producer, which cannot meet the demand using only one unit of labor, say. The question arises whether one can revive the viability of the allocation mechanism by increasing the number of labor units : We introduce for that purpose a scalar** $w \in [1, +\infty[$ **measuring the number of labor units and consider the new set of allocations defined by**

$$K \ := \ \left\{ (x, y, w) \in \prod_{i=1}^{n} L_i \times M \times [1, +\infty[ \ \mid \ \sum_{i=1}^{n} x_i \leq wy \right\} \tag{10}$$

**Assume that the autonomous dynamics of the consumers $i$ are defined by $g_i : Y \mapsto Y$, that the autonomous dynamics of the producer is described by $g_0 : Y \mapsto Y$ and that the autonomous dynamics governing the number of labor units is given by $e : \mathbf{R} \mapsto \mathbf{R}$ (a reasonable choice is to take $e = 0$). We derive that *the constraints involving labor units are viable under "dynamical economies" of the form***

$$\begin{cases} x_i'(t) &= g_i(x_i(t)) - p(t) \quad (i = 1, \ldots, n) \\ y'(t) &= g_0(y(t)) + w(t)p(t) \\ w'(t) &:= e(w(t)) + \langle p(t), y(t) \rangle \end{cases}$$

**where $p(t) \in Y^\star$ is a price regulating the evolution by slowing down the consumptions of the consumer, increasing the production of the producer and governing the evolution of the labor units in terms of the income $\langle p(t), y(t) \rangle$ they receive from the production at each instant.** □

**Using** *viability multipliers*, **one can prove that dynamical systems of the form**

$$\begin{cases} x_0'(t) \; = \; g_0(x_0(t)) - W_0^{\star}(t)p_1(t) & (j=0) \\ x_j'(t) \; = \; g_j(x_j(t)) + p_j(t) - W_j^{\star}(t)p_{j+1}(t) & (j=1,\ldots,m-1) \\ x_m'(t) \; = \; g_m(x_m(t)) + p_m(t) & (j=m) \\ W_j'(t) \; = \; e(W_j(t)) - x_j(t) \otimes p_{j+1}(t) & (j=0,\ldots,m-1) \end{cases}$$

**govern viable solutions provided that the evolution of technological matrices** $W_j(t)$ **obeys dynamics involving the tensor product of** $x_j(t)$ **and the "intermediate price"** $p_{j+1}(t)$**.** □

**Let** $X_j$**,** $Y_j$ **,** $Z_j$ **(**$j = 0, \ldots m-1$**) and** $Y_j$ **(**$j = i, \ldots m$**) denote finite dimensional spaces.**

**Consider linear operators** $A_j \in \mathcal{L}(X_j, Y_j)$**,** $B_j \in \mathcal{L}(Z_j, X_{j+1})$ **and** $C_j \in \mathcal{L}(X_j, Y_j)$ **and "technological matrices"** $W_j \in \mathcal{L}(Y_j, Z_j)$**.**

**The hierarchical process takes the form**

$$\forall\, j = 1, \ldots m, \;\; B_{j-1}W_{j-1}A_{j-1}x_{j-1} + C_j x_j \;\in\; M_j$$

**where** $M_j \subset Y_j$ **are closed convex subsets.**

**Here** $x_0 \in X_0$ **is regarded as an output,** $x_m \in X_m$ **as an input, whereas** $x_j \in X_j$ **is regarded either as intermediate outputs or inputs.**

**Observe that taking $M_j := \{0\}$ for $j = 1, \ldots, m-1$ and $C_j := -\mathbf{1}$ for $j = 1, \ldots, m$, the above constraints read:**

$$B_{m-1}W_{m-1}A_{m-1} \cdots B_{j-1}W_{j-1}A_{j-1} \cdots B_0W_0A_0x_0 - x_m \ \in \ M_m$$

**Taking $M_j := \{0\}$ for $j = 1, \ldots, m-1$ and $C_j := -\mathbf{1}$ for $j = 1, \ldots, m-1$ and $C_m = 0$, we find constraints of the form**

$$B_{m-1}W_{m-1}A_{m-1} \cdots B_{j-1}W_{j-1}A_{j-1} \cdots B_0W_0A_0x_0 \ \in \ M_m$$

**Therefore, in a hierarchical process, the state variables are the sequences**

$$((x_j)_{j=0,\ldots,m},\ (W_j)_{j=0,\ldots,m-1})$$

**made of vectors and of matrices. The above constraints are defined by inverse images under a multilinear map.**

$$\left\{\begin{array}{ccc} & A_{j-1} & \\ x_{j-1} \in X_{j-1} & \longmapsto & Y_{j-1} \\ & W_{j-1} & \downarrow \\ & & Z_{j-1} \\ & B_{j-1} & \downarrow \\ x_j \in X_j & \longmapsto & Y_j \supset M \\ & C_j & \end{array}\right.$$

**However, the linearity of the operators $A_j,\ B_j$ & $C_j$ does not really bring any simplification in the viability analysis. This is the reason why we study this example in the more general nonlinear form described in the following way:**

**1.** $x_j \in L_j \subset X_j$, $(j = 1, \ldots m)$,

2. **continuously differentiable maps** $l_j : X_j \mapsto Y_j$ **and** $h_j : Z_{j-1} \times X_j \mapsto Y_j$**,**

3. **"technological matrices"** $W_j \in \mathcal{W}_j \subset \mathcal{L}(Y_j, Z_j)$

4. **"scarcity constraint"** $h_j(W_{j-1} l_{j-1}(x_{j-1}), x_j) \in M_j \subset Y_j$

**The constrained set is defined by**

$$\begin{array}{l} K \ := \ \{((x_j)_{j=0,\ldots,m}, \ (W_j)_{j=0,\ldots,m-1}) \in \prod_{j=0}^{m} L_j \times \prod_{j=0}^{m-1} \mathcal{W}_j \\ \textbf{\textit{such that}}\, h_j(W_{j-1} l_{j-1}(x_{j-1}), x_j) \ \in \ M_j \} \end{array} \tag{11}$$

**We set**

$$A_j \ := \ l'_j(x_j), \ \ B_{j-1} \ := \ h'_{j_{z_{j-1}}}(W_{j-1} l_{j-1}(x_{j-1}), x_j)$$

**and**

$$C_j \ := \ h'_{j_{x_j}}(W_{j-1} l_{j-1}(x_{j-1}), x_j)$$

**Hence, the directional derivative is equal to**

$$\left\{ \begin{array}{l} h'_j(W_{j-1} l_{j-1}(x_{j-1}), x_j)(u_{j-1}, u_j, V_{j-1}) \\ = \ C_j u_j + B_{j-1} W_{j-1} A_{j-1} u_{j-1} + B_{j-1} V_{j-1} l_{j-1}(x_{j-1}) \end{array} \right.$$

**and can be written in the form**

$$B_{j-1} W_{-1} A_{j-1} u_{j-1} + C_j u_j + (l_{j-1}(x_{j-1}) \otimes B_{j-1}) V_{j-1}$$

**The transpose of this operator maps $(q_j) \in \prod_{j=1}^m Y_j^\star$ to**

$$(X_0 \times \mathcal{L}(Y_0^\star, Z_0^\star)) \times \left( \prod_{j=1}^{m-1} (X_j^\star \times \mathcal{L}(Y_j^\star, Z_j^\star) \right) \times X_m^\star$$

**in the following way:**

$$\left\{ \begin{array}{cl} (A_0^\star W_0^\star B_0^\star q_1 \ , \ l_0(x_0) \otimes B_0^\star q_1) & (j = 0) \\ (A_j^\star W_j^\star B_j^\star q_{j+1} + C_j^\star q_j \ , \ l_j(x_j) \otimes B_j^\star q_{j+1}) & (j = 1, \ldots, m-1) \\ C_m^\star q_m & (j = m) \end{array} \right.$$

**We posit a sufficient condition guaranteeing the transversality conditions:**

$$\forall \ ((x_j)_{j=0,\ldots,m}, \ (W_j)_{j=0,\ldots,m-1}), \ \ h'_{j_{x_j}}(W_{j-1} l(x_{j-1}), x_j) \ \text{ are surjective} \tag{12}$$

**Assume now that the dynamical behaviors of the autonomous systems are described by**

$$\left\{ \begin{array}{rcll} x_j'(t) & = & g_j(x_j(t))(t) & (j = 0, \ldots m) \\ \\ W_j'(t) & = & e(W_j(t)) & (j = 0, \ldots m-1) \end{array} \right.$$

**where $y_j(t) \in Y_j$ and $q_j(t) \in Z_j^\star$.**

**We deduce that *the constraints (11) involving connection matrices are viable under control systems of the form***

$$\begin{cases} x_0'(t) \ = \ g_0(x_0(t)) - A_0^\star W_0^\star(t) B_0^\star p_1(t) \\ \\ x_j'(t) \ = \ g_j(x_j(t)) - C_j^\star p_j(t) - A_j^\star W_j^\star(t) B_j^\star p_{j+1}(t) \ \ (j = 1, \ldots, m-1) \\ \\ x_m'(t) \ = \ g_m(x_m(t)) - C_m^\star p_m(t) \\ \\ W_j'(t) \ := \ e(W_j(t)) - l_j(x_j(t)) \otimes B_j^\star p_{j+1}(t) \ \ (j = 0, \ldots, m-1) \end{cases}$$

**The regulation map can be written in the form in the following way:** $(p_j)_{j=1,\ldots,m}$ **belongs to** $\Pi_K((x_j)_{j=0,\ldots,m}, \ (W_j)_{j=0,\ldots,m-1})$ **if and only if** $p$ **is a solution to the system of equations**

$$\begin{cases} (C_1 C_1^\star + B_0 W_0 A_0 A_0^\star W_0^\star B_0^\star)\, p + \langle p, B_0 l_0(x_0) \rangle B_0 l_0(x_0) + C_1 A_1^\star W_1^\star B_1^\star p_2 \ = \ z_1 \\ \\ \left(C_j C_j^\star + B_{j-1} W_{j-1} A_{j-1} A_{j-1}^\star W_{j-1}^\star B_{j-1}^\star\right) p_j \\ + \langle p_j, B_{j-1} l_{j-1}(x_{j-1}) \rangle B_{j-1} l_{j-1}(x_{j-1}) + C_j A_j^\star W_j^\star B_j^\star p_{j+1} \\ + B_{j-1} W_{j-1} A_{j-1} C_{j-1}^\star p_{j-1} \ = \ z_j \ \ (j = 2, \ldots, m-1) \\ \\ (C_m C_m^\star + B_{m-1} W_{m-1} A_{m-1} A_{m-1}^\star W_{m-1}^\star B_{m-1}^\star)\, p_m \\ + \langle p_m, B_{m-1} l_{m-1}(x_{m-1}) \rangle B_{m-1} l_{m-1}(x_{m-1}) + B_{m-1} W_{m-1} A_{m-1} C_{m-1}^\star p_{m-1} \ = \ z_m \end{cases}$$

**where where $r_j$ belongs to $T_{M_j}(h_j(W_{j-1}l(x_{j-1}), x_j))$ and where**

$$z_j \ := \ C_j g(x_j) + B_{j-1}W_{j-1}A_{j-1}g_{j-1}(x_{j-1}) + B_{j-1}e_{j-1}(W_{j-1})l(x_{j-1}) - r_j$$

**In other words, this solution is obtained by inverting a tri-diagonal block matrix:**

$$\begin{pmatrix}
\mathcal{J}_1 & \mathcal{M}_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\mathcal{M}_1^\star & \mathcal{J}_2 & \mathcal{M}_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \mathcal{M}_{j-1}^\star & \mathcal{J}_j & \mathcal{M}_j & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\mathcal{M}_{m-2}^\star & \mathcal{J}_{m-1} & \mathcal{M}_{m-1} \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \mathcal{M}_{m-1}^\star & \mathcal{J}_m
\end{pmatrix}$$

**where we set**

$$\left\{ \begin{array}{l} \mathcal{J}_j p_j \\ := \ \left(C_j C_j^\star + B_{j-1}W_{j-1}A_{j-1}A_{j-1}^\star W_{j-1}^\star B_{j-1}^\star\right) p_j + \langle p_j, B_{j-1}l_{j-1}(x_{j-1})\rangle B_{j-1}l_{j-1}(x_{j-1}) \end{array} \right.$$

**and**

$$\mathcal{M}_j \ := \ C_j A_j^\star W_j^\star B_j^\star$$