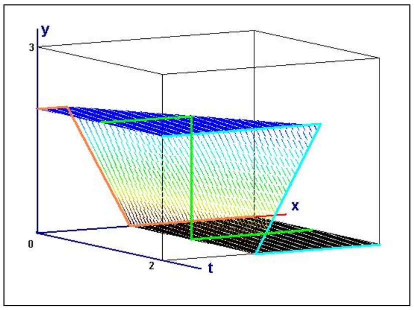
Burgers' Equations

$$\frac{\partial \mathbf{U}(t,x)}{\partial t} + \frac{\partial \mathbf{U}(t,x)}{\partial x}\mathbf{U}(t,x) \;=\; 0$$



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Outline

Motivations

We shall motivate in this presentation the Burgers' equation as a prototype of a general "tracking problem".

Consider an evolution $x(t) \in X := \mathbb{R}$.

Knowing that its evolution is governed by a second-order differential equation $x''(t) = g(t, x(t), x'(t))$ or a second-order differential inclusion $x''(t) \in G(t, x(t), x'(t))$, we are looking for a (set-valued) map $(t, x) \in \mathbb{R}_+ \times X \rightsquigarrow \mathbf{U}(t, x) \in X$ tracking the velocities of $x'(t) \in X$ of $x(t) \in X$ at time $t \geq 0$ in the sense that

$$\forall\, t \geq 0,\ \ x'(t)\ \in\ \mathbf{U}(t, x(t))$$

We assume that initial time $t = 0$, the initial velocities $y \in \mathbf{U}_0(x)$ at each $x \in X$ are known: The set-valued map $\mathbf{U}_0 : X \rightsquigarrow Y$ is a given initial condition for the tracking problem.

The Tracking Problem

Let us fix a given time T and a position x. If the evolution of velocities $y(\cdot)$ is known, then the evolution of $x(t)$ *passing through* $x(\cdot)$ *at time* T is given by

$$\forall\, t \geq 0,\;\; x(t) \;=\; x - \int_t^T y(\tau)d\tau$$

We are actually looking for a set-valued map $\mathbf{U} : \mathbb{R}_+ \times X \rightsquigarrow Y$ providing the velocities $y \in \mathbf{U}(T,x)$ such that there exists an evolution $y(\cdot)$ of the velocities satisfying

$$\left\{\begin{array}{ll} (i) & y(T) \;=\; y \\ (ii) & \forall\, t \geq 0,\;\; y(t) \;\in\; \mathbf{U}\left(t, x - \displaystyle\int_t^T y(\tau)d\tau\right) \\ (iii) & y(0) \;\in\; \mathbf{U}_0\left(x - \int_0^T y(\tau)d\tau\right) \end{array}\right.$$

Condition (ii) amounts to saying that $y(t) \in \mathbf{U}(t, x(t))$, i.e., that the velocities “tracks” $x(t)$ at each time.

Burger's Tracking Problem

For the Burgers tracking problem, the acceleration is assumed to be equal to zero, or the velocities to be constant:

Definition 1 ***Given the initial repartition*** $\mathbf{U}_0$, ***find the set-valued map*** $\mathbf{U} : \mathbb{R}_+ \times X \rightsquigarrow Y$ ***such that***

$$\forall\, t \geq 0, \ \ y \ = \ \mathbf{U}\,(t, x + (t - T)y)$$

satisfying the initial condition

$$\forall\, x \in X, \ \ \mathbf{U}(0, x) \ = \ \mathbf{U}_0(x)$$

Theorem 4 below states that *there exists a unique solution* to this Burgers tracking problem: $\mathbf{U}(T, x)$ **is the set of fixed points** $y \in \mathbf{U}_0(x - Ty)$ **of the set-valued map** $y \rightsquigarrow \mathbf{U}_0(x - Ty)$**.**

Set-Valued Solutions

Consequently, even when the initial condition U_0 is single-valued, the subset $\mathrm{U}(T,x)$ of fixed points may be set-valued for several values of T and x. The following cases do appear:

1. **$\mathrm{U}(t,x) = \{y\}$ (singleton)**
2. **$\mathrm{U}(t,x) = \{y_1, \ldots, y_p\}$ (finite number of branches)**
3. **$\mathrm{U}(t,x) = S$ where S is an interval (shock)**
4. **$\mathrm{U}(t,x) = \emptyset$ (the solution ceases to exist)**

These phenomena (shocks, several branches, nonexistence) obtained even for the simplest initial conditions have been observed since Riemann, and have embarrassed physicists (who hate to have set-valued solutions and demand to single out THE single-valued ONE which makes "physical sense".

Burgers Type Partial Differential Equations

If the solution $\mathbf{U}(t,x)$ is single-valued and differentiable, then we can formally differentiate the tracking property

$$\forall\, t \geq 0, \;\; y(t) \;=\; \mathbf{U}\left(t, x - \int_t^T y(\tau)d\tau\right)$$

and derive

$$y'(t) \;=\; \frac{\partial \mathbf{U}(t,x(t))}{\partial t} + \frac{\partial \mathbf{U}(t,x(t))}{\partial x} y(t) \;=\; \frac{\partial \mathbf{U}(t,x(t))}{\partial t} + \frac{\partial \mathbf{U}(t,x(t))}{\partial x} \mathbf{U}(t,x(t))$$

If the evolution of velocities $y(t) := x'(t)$ is governed by $y'(t) = g(t,x(t),y(t))$, $\mathbf{U}$ is a solution to the partial differential equation

$$\frac{\partial \mathbf{U}(t,x)}{\partial t} + \frac{\partial \mathbf{U}(t,x)}{\partial x} \mathbf{U}(t,x) \;=\; g(t,x,\mathbf{U}(t,x))$$

and when $g \equiv 0$, to the Burgers partial differential equation

$$\frac{\partial \mathbf{U}(t,x)}{\partial t} + \frac{\partial \mathbf{U}(t,x)}{\partial x} \mathbf{U}(t,x) \;=\; 0 \qquad (1)$$

satisfying the initial condition $\mathbf{U}(0,x) = \mathbf{U}_0(x)$.

Set-Valued Solutions to PDE's

However, the solutions to the Burgers tracking problem may be set-valued.

This is not really a problem, since we can differentiate set-valued maps thanks to the concept of contingent derivatives of set-valued maps and give a meaning to the above nonlinear partial differential equations.

We will do that even when the velocities are governed by a differential inclusion $y'(t) \in G(t, x(t), y(t))$ and U is a solution to a partial differential inclusion

$$\frac{\partial \mathbf{U}(t,x)}{\partial t} + \frac{\partial \mathbf{U}(t,x)}{\partial x}\mathbf{U}(t,x) \ \in \ G(t,x,\mathbf{U}(t,x))$$

Actually, we shall show that the "viability solution" to any tracking problem that we shall define is the unique solutions (in the Frankowska sense) to partial differential inclusions satisfying initial/boundary conditions of various kinds (See Theorem 12 below).

Furthermore, we shall prove that the graphs of viability solutions to tracking problems are viability kernels, and thus inherit their properties.

They can be computed by the Saint-Pierre Viability Kernel Algorithm. So, we shall derive them in this chapter, and not from the fact that they are solutions to partial differential equations or inclusions.

Outline

Capture Basin of a Target

Definition 2 ***Let*** $K \subset X$ ***be a constrained set and*** $C \subset K$ ***be a target.***

The subset $\mathrm{Capt}_F(K, C)$ ***of initial states*** $x_0 \in K$ ***such that at least one evolution*** $x(\cdot) \in \mathcal{S}(x_0)$ ***starting at*** x_0 ***to the differential inclusion*** $x'(t) \in F(x(t))$ ***is viable in*** K ***until it reaches*** C ***in finite time is called the* capture basin of** C **viable in** K ***under*** $\mathcal{S}$***.***

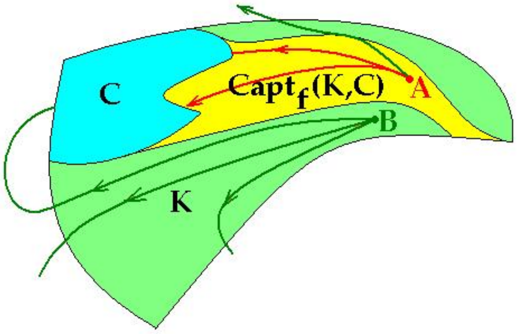

Figure 1: **Capture Basin of a Target**

Definition of the Viability Solution

Definition 3 ***We define*** $\mathbf{U}_0^+ : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ ***by***

$$\mathbf{U}_0^+(t,x) \;=\; \begin{cases} \mathbf{U}_0(x) & \text{if} \quad t \;=\; 0 \;\&\; x \in \mathbb{R} \\ \emptyset & \text{if} \quad t \;\neq 0 \;\&\; x \in \mathbb{R} \end{cases}$$

We observe that $\text{Graph}(\mathbf{U}_0^+) = \{0\} \times \text{Graph}(\mathbf{U}_0)$***.***

Let us introduce the "characteristic system"

$$\begin{cases} (i) & \tau'(t) \;=\; -1 \\ (ii) & x'(t) \;=\; -y \\ (iii) & y'(t) \;=\; 0 \end{cases} \tag{2}$$

We shall say that the set-valued map $\mathbf{U} : \mathbb{R}_+ \times \mathbb{R}_+ \rightsquigarrow \mathbb{R}$ ***defined by***

$$\text{Graph}(\mathbf{U}) \;:=\; \text{Capt}_{(2)}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \text{Graph}(\mathbf{U}_0^+)) \tag{3}$$

is the **viability solution** ***to the Burgers Tracking Problem (1) satisfying the initial condition*** $\mathbf{U}(0,x) := \mathbf{U}_0(x)$***.***

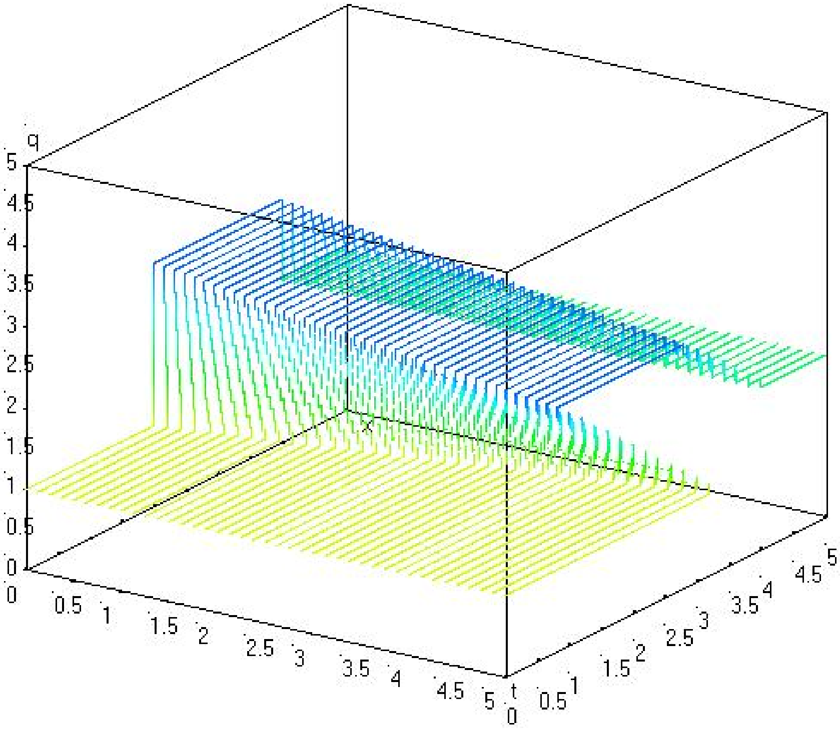

Figure 2: **Graph of the Viability Solution starting from a set-valued map initial condition**. Source: Patrick Saint-Pierre.

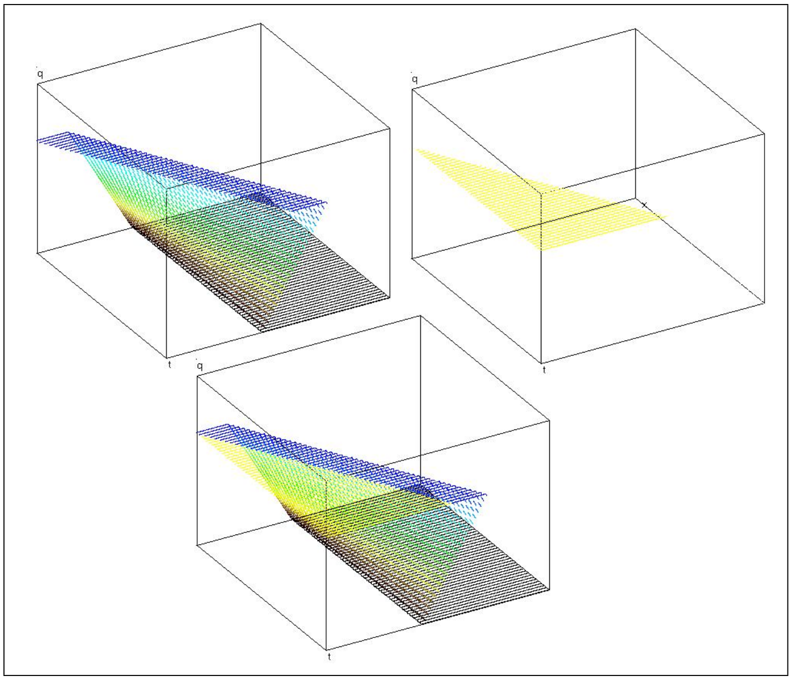

Figure 3: Viability Kernel and Inert Evolution. By clicking on Figure 3, on box 'Evaluation' in the upper-left, by choosing whether you want the solution for either the initial condition only, the boundary condition only or both, you will see the graph of the initial condition, contained is the plane $t = 0$, and the graph of the boundary condition, contained in union of the two planes $x = 0$ (beginning of the interval $[0, 5]$ and the plane $x = 5$. By clicking on box 'Show, and waiting for a few seconds, the Saint-Pierre Viability Kernel Algorithm computes the capture basin of the of the graphs of the initial and/or boundary conditions, which is the graph of the viability solution $\mathbf{U}$ to the initial and/or boundary-value Burgers' problem. Before running the programme again, clock on box 'Erase'. The solutions for either initial value problem only or boundary-value problem only have empty values. See Figures 7 and 8. Thanks to the morphism property of the viability solution (the solution of an union of condition is the union of the solutions), we check numerically that the viability solution of the initial/boundary-value Burgers' problem (lower center) is the union of the solution of the initial-value problem (upper left) and of the solution to the boundary-value problem (upper right). Programme by Patrick Saint-Pierre.

Existence and Uniqueness Theorem

Theorem 4 *The viability solution* $\mathbf{U}$ *is the* unique *solution* $\mathbf{V}$ *to the Burger tracking problem (1).*

Furthermore, $\mathbf{U}(t,x)$ *is the set of fixed point* $y \in \mathbf{U}_0(x-ty)$ *of the map* $y \rightsquigarrow \mathbf{U}_0(x-ty)$*.*

The viability solution satisfies the "maximum principle":

$$\forall\,(t,x) \in \mathbb{R}_+ \times X,\quad \sup_{y \in \mathbf{U}(t,x)} |y| \;\leq\; \sup_{x \in X}\, \sup_{y \in \mathbf{U}_0(x)} |y|$$

or, more precisely

$$\forall\,(t,x) \in \mathbb{R}_+ \times X,\quad \mathbf{U}(t,x) \;\subset\; \mathrm{Im}(\mathbf{U}_0)$$

The inverse $y \mapsto \mathbf{U}^{-1}(t,y)$ *of the solution map* $x \mapsto \mathbf{U}(t,x)$ *is given by*

$$\mathbf{U}^{-1}(t,y) \;=\; \mathbf{U}_0^{-1}(y) + ty$$

Fixed Point Theorems

The above formula requires fixed-point theorems to guarantee the non-emptiness of $\mathbf{U}(T, x)$.

The Brouwer Fixed Point Theorem states that if the image $\mathrm{Im}(\mathrm{U}^{\mathbf{0}}) \subset [a, b]$ is bounded and if $\mathbf{U}_0$ is single-valued and continuous, then there exists a fixed point on the compact interval $[a, b]$.

If $\mathbf{U}_0$ is set-valued map, if its graph is closed, if image $\mathrm{Im}(\mathrm{U}^{\mathbf{0}}) \subset [a, b]$ is bounded and if the images are convex, the Kakutani Fixed Point Theorem, an extension of the Brouwer Fixed Point Theorem, guarantees also the existence of a fixed point of the set-valued map $y \rightsquigarrow \mathbf{U}_0(x - Ty)$.

The Banach-Picard Contraction Theorem implies existence and uniqueness of a fixed point of $y \rightsquigarrow \mathbf{U}_0(x - Ty)$ if $\mathbf{U}_0$ is single-valued and Lipschitz with constant λ and if $T < \frac{1}{\lambda}$ (because the map $y \mapsto \mathbf{U}_0(x - Ty)$ is Lipschitz with constant $T\lambda < 1$).

Single-Valuedness of the Viability Solution

Assume that the initial condition $\mathrm{U}_0 : \mathbb{R} \rightsquigarrow \mathbb{R}$ is monotone (increasing) in the sense that there exists a constant $c \in \mathbb{R}$ (positive or negative) such that

$$\forall\, (x_1, x_2),\ \forall\, y_1 \in \mathbf{U}_0(x_1),\ y_2 \in \mathbf{U}_0(x_2),\ \ (y_1 - y_2)(x_1 - x_2) \geq\ c(x_1 - x_2)^2$$

Then the solution $\mathrm{U}(t, \cdot)$ to the Burgers equation starting at U_0 is single-valued whenever $t \geq 0$ if $c \geq 0$ and $0 \leq t < \frac{1}{|c|}$ if $c < 0$.

Outline

1. *Introduction*
2. *Viability Solutions for Initial Value Problems*
3. ▶ *Morphism Property of the Viability Solution*
4. *Piecewise Linear Initial Conditions*
5. *Viability Solutions under Viability Constraints and/or Boundary Conditions*
6. *Derivatives of Set-Valued maps and Frankowska Solutions to the Burgers Equation*
7. *Regulating Controlled Burgers Tracking Problem*

Union Dependence on Initial Conditions

The capture basin of an union of targets being (obviously) the union of the capture basins of each of the targets, the map associating with any initial condition $\mathbf{U}_{0_i}(x)$ the solution $\mathbf{U}_i(t,x)$ is a *morphism with respect to the union* (of set-valued maps):

$$\text{if } \mathbf{U}_0(x) := \bigcup_{i\in\mathbb{I}} \mathbf{U}_{0_i}(x), \text{ then } \mathbf{U}(t,x) = \bigcup_{i\in\mathbb{I}} \mathbf{U}_i(t,x)$$

The solution depends “unionly” on the initial conditions (instead of linearly).

The group structure $(+,0)$ of the vector space is replaced by the lattice structure $(\cup,\emptyset)$ on the subsets of the vector space, for which the maps associating an initial condition the solution of the semi-linear equation is a lattice-morphism.

This morphism property is as useful as the linearity property of solutions to linear systems.

Continuous Dependence on Initial Conditions

We also derive from the stability properties of capture basins that the solution to the Burgers equations *depends continuously on the initial conditions* for an adequate concept of convergence:

Since the initial conditions and/or the solutions may be set-valued maps, this stability property has a meaning when the *convergence of the set-valued maps is defined by the convergence of their graphs* (graphical convergence).

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Characteristic Set-Valued Maps of Sets

The characteristic set-valued map Ξ_A of a subset $A \subset X$ of any vector space is defined by

$$\Xi_A(x) \;:=\; \Xi(A;x) \;:=\begin{cases} 1 & \textbf{if} \quad x \in A \\ \emptyset & \textbf{if} \quad x \notin A \end{cases}$$

If $F : X \rightsquigarrow Y$ is a set-valued map, we denote by $F\Xi_A : X \rightsquigarrow Y$ the set-valued map defined by

$$F(x)\Xi_A(x) \;:=\; F(x)\Xi(A;x) \;:=\begin{cases} F(x) & \textbf{if} \quad x \in A \\ \emptyset & \textbf{if} \quad x \notin A \end{cases}$$

In particular, the *shock at a point σ of intensity $S \subset Y$* is described by $S\Xi_\sigma$ and associates with any x the subset S when $x = \sigma$ and the empty set otherwise.

Piecewise Linear Initial Conditions

We use the remarkable morphism property to compute the solutions starting at initial condition

$$\mathbf{U}_0(x) \ = \bigcup_{i=0}^{n} (\alpha_i x + \beta_i)\Xi_{A_i}(x)$$

Observe that if the $n+1$ intervals A_i associated with a finite sequence $\delta_1 < \ldots < \delta_n$ by formulas

$$\begin{cases} A_0 :=]-\infty, \delta_1] \\ A_i :=]\delta_i, \delta_{i+1}], \quad i = 1, \ldots, n-1 \\ A_n :=]\delta_n, +\infty[\end{cases}$$

form a *partition* of $\mathbb{R}$, then

$$\mathbf{U}_0(x) \ = \sum_{i=0}^{n} (\alpha_i x + \beta_i)\chi_{A_i}(x)$$

where the functions χ_{A_i} are the usual characteristic functions.

Some Building Block Solutions

1. **If** $\mathbf{U}_0 := 0\Xi_A$**, then,** $\mathbf{U}(t,x) = 0\Xi(A;x)$**, so that the these maps are invariant under the Burgers equation along time.**
2. **If** $\mathbf{U}_0 := \beta\Xi_A$**, then** $\mathbf{U}(t,x) = \beta\Xi(A+\beta t;x)$**.**
3. **If** $\mathbf{U}_0 := (\alpha x+\beta)\Xi_A(x)$**,then**
    - **If** $t \neq -\frac{1}{\alpha}$**, then** $\mathbf{U}(t,x) := \left(\dfrac{\alpha x+\beta}{1+\alpha t}\right)\Xi((1+\alpha t)A+\beta t;x)$
    - **If** $t = -\frac{1}{\alpha}$**, then there exists a shock of size** $\alpha A+\beta$ **at** $-\dfrac{\beta}{\alpha}$**:** $\mathbf{U}\left(-\dfrac{1}{\alpha},x\right) := (\alpha A+\beta)\,\Xi\left(-\dfrac{\beta}{\alpha};x\right)$**.**
4. **If** $\mathbf{U}_0(x) = S\Xi_\sigma(x)$ **is a chock of size** S **at** $x=\sigma$**, then** $\mathbf{U}(t,x) = \left(\dfrac{x-\sigma}{t}\right)\Xi\left(tS+\sigma;x\right)$**.**

Proposition 5 ***The viability solution to the Burgers tracking problem (1) satisfying the initial condition***

$$\mathbf{U}(0,x) \ = \bigcup_{i\in I}(\alpha_i x+\beta_i)\Xi_{A_i}(x)$$

is equal to:

- ***Case when*** $t \neq -\frac{1}{\alpha_i}$ ***for all*** $i \in \mathbb{I}$***,***

$$\mathbf{U}(t,x) \ = \bigcup_{i\in\mathbb{I}}\left(\frac{\alpha_i x+\beta_i}{1+\alpha_i t}\right)\Xi\left((1+\alpha_i t)A_i+t\beta_i;x\right) \tag{4}$$

The cardinal of the set $\mathbb{I}(t,x) \ := \ \{i\in\mathbb{I}$ ***such that*** $x\in(1+\alpha_i t)A_i+t\beta_i\}$ ***denotes the number of elements of*** $\mathrm{U}(t,x)$ ***and plays the role of a "*****valuemeter*****" since we actually can also write***

$$\mathbf{U}(t,x) \ = \ \left\{\frac{\alpha_i x+\beta_i}{1+\alpha_i t}\right\}_{i\in\mathbb{I}(t,x)}$$

- ***Case when $\alpha_i < 0$ and $t = -\frac{1}{\alpha_i}$ for some $i \in \mathbb{I}$: we obtain shocks:***

$$\mathbf{U}\left(-\frac{1}{\alpha_i}, x\right) \;=\; (\alpha_i A_i + \beta_i)\,\Xi\left(-\frac{\beta_i}{\alpha_i}; x\right)$$

at points $-\dfrac{\beta_i}{\alpha_i}$ of size $\alpha_i A_i + \beta_i$, which plays the role of a "**valuemeter*****" in case of shocks because we can write***

$$\mathbf{U}\left(-\frac{1}{\alpha_i}, -\frac{\beta_i}{\alpha_i}\right) \;=\; \alpha_i A_i + \beta_i$$

Examples

We now provide examples of more specific (and classical) initial conditions by applying Proposition 5 and the particular solutions already provided:

In particular, we obtain the following "building blocks" for constructing solutions to the Burgers equation with piecewise linear initial conditions:

1. **O-Characteristic set-valued maps:**

$$\begin{cases} \text{if } = \mathbf{U}_0 := 0\Xi_{]-\infty,0]} & \text{then} \quad \mathbf{U}(t,x) = 0\Xi(]-\infty,0];x) \\ \text{if } = \mathbf{U}_0 := 0\Xi_{[0,\infty[} & \text{then} \quad \mathbf{U}(t,x) = 0\Xi([0,+\infty[;x) \\ \text{if } = \mathbf{U}_0 := 0\Xi_{]-\infty,1]} & \text{then} \quad \mathbf{U}(t,x) = 0\Xi(]-\infty,1];x) \\ & \end{cases}$$

2. **Characteristic set-valued maps:**

$$\begin{cases} \text{if } = \mathbf{U}_0 := \Xi_{]-\infty,0]} & \text{then} \quad \mathbf{U}(t,x) = \Xi(]-\infty,t];x) \\ \text{if } = \mathbf{U}_0 := \Xi_{[0,\infty[} & \text{then} \quad \mathbf{U}(t,x) = \Xi([t,+\infty[;x) \\ \text{if } = \mathbf{U}_0 := \Xi_{[1\infty[} & \text{then} \quad \mathbf{U}(t,x) = \Xi([t+1,+\infty[;x) \\ & \end{cases}$$

3. Affine Maps

- **Increasing Linear Maps**

$$\text{if } = \mathbf{U}_0 := x\Xi_{[0,1]} \ \text{ then } \ \mathbf{U}(t,x) = \frac{x}{1+t}\Xi([0,1+t];x)$$

- **Decreasing Linear Maps and Emergence of Shocks**

$$\text{If } \mathbf{U}_0(x) = (1-x)\Xi_{[0,1]}, \ \text{then } \mathbf{U}(t,x) \ = \begin{cases} \frac{1-x}{1-t} & \text{if } \ \min(t,1) \leq x < 1 \\ [0,1] & \text{if } \ x = 1 \ \& \ t = 1 \text{ (shock)} \\ \frac{x-1}{t-1} & \text{if } \ 1 < x \leq \max(t,1) \end{cases}$$

4. Shocks

$$\text{if } = \mathbf{U}_0 := [0,1]\,\Xi_0 \ \text{ then } \ \mathbf{U}(t,x) = \left(\frac{x-1}{t}\right)\Xi([1,1+t];x)$$

By combining them, we obtain

1. **If the initial condition $\mathrm{U}_0 := \xi_A = \Xi_A \cup 0\Xi_{\complement(A)}$ is the characteristic function A defined by**

$$\mathbf{U}_0(x) \;:= \begin{cases} 1 & \textbf{if} \quad x \in A \\ 0 & \textbf{if} \quad x \notin A \end{cases}$$

   the solution U is given by

$$\mathbf{U}(t,x) \;=\; \Xi(A+t;x) \cup \Xi(\complement(A);x) \;=\; \begin{cases} 1 & \text{if} \quad x \;\in\; A \cap (A+t) \\ 0 & \text{if} \quad x \in \complement(A \cup (A+t)) \\ \{0,1\} & \text{if} \quad x \in (A+t) \setminus A \\ \emptyset & \text{if} \quad A \setminus (A+t) \end{cases}$$

2. If the initial condition is $\mathbf{U}_0 := \Xi_{]-\infty,0]} \cup 0\Xi_{[0,+\infty[}$ defined by

$$\mathbf{U}_0(x) \;:=\; \left\{ \begin{array}{lll} 1 & \textbf{if} & x \leq 0 \\ 0 & \textbf{if} & x \geq 0 \end{array} \right.$$

(Riemann's Problem), then the viability solution is given by the formula:

$$\mathbf{U}(t,x) \;=\; \Xi([-\infty,t]\,;x) \cup 0\Xi([0,+\infty[;x) \;=\; \left\{ \begin{array}{lll} 1 & \text{if} & x \;\leq\; 0 \\ \{0,1\} & \text{if} & 0 \leq x \leq t \\ 0 & \text{if} & x \geq t \end{array} \right.$$

and the viability solution is set-valued : its images contains two points for $x \in [0,t]$. (see Figure 4).

3. If the initial condition is $\mathbf{U}_0 = 0\Xi_{]-\infty,0]} \cup \Xi_{[0,+\infty[}$ **defined by**

$$\mathbf{U}_0(x) \; := \begin{cases} 0 & \textbf{if} \quad x \leq 0 \\ 1 & \textbf{if} \quad x \geq 0 \end{cases}$$

then the viability solution becomes

$$\mathbf{U}(t,x) \; = \; \Xi(]-\infty,0];x) \cup \Xi([0,+\infty[;x) \; = \begin{cases} 0 & \text{if} \quad x \; \leq \; 0 \\ \emptyset & \text{if} \quad 0 \leq x \leq t \\ 1 & \text{if} \quad x \geq t \end{cases}$$

and the viability solution is set-valued : its images are empty for $x \in [0,t]$ **(see Figure 5).**

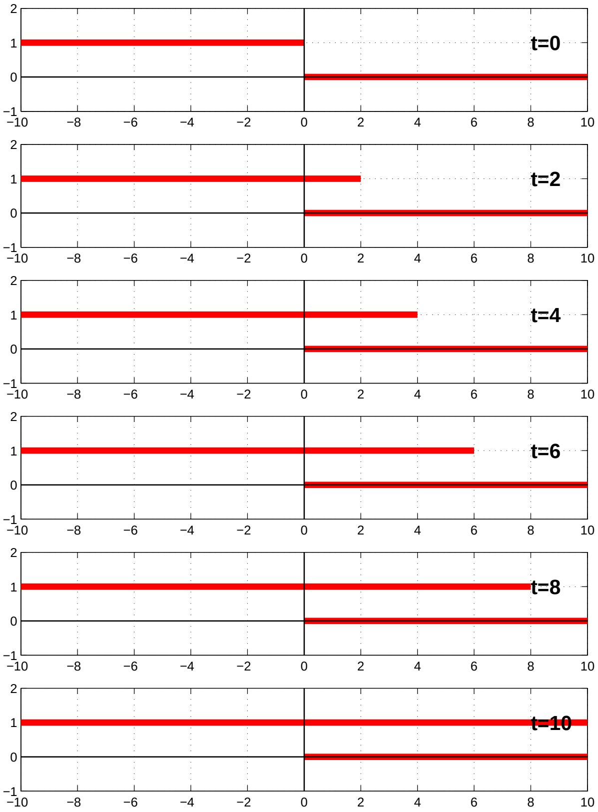

Figure 4:

4. **If the initial condition is** $\mathrm{U}_0 := 0\Xi_{]-\infty,1]} \cup [0,1]\,\Xi_1 \cup \Xi_{[1,+\infty[}$ **describing a shock at time** 0 **at** $x = 1$**, given by**

$$\mathbf{U}_0(x) \;=\; \begin{cases} 0 & \text{if} \quad x \leq 1 \\ [0,1] & \text{if} \quad x = 1 \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

we obtain

$$\mathbf{U}(t,x) = \Xi(]-\infty,1];x) \cup \left(\frac{x-1}{t}\right)\Xi([1,1+t];x) \cup \Xi([1+t,+\infty[;x) \;=\; \begin{cases} 0 & \text{if} \quad x \leq 1 \\ \frac{x-1}{t} & \text{if} \quad 1 < x < 1+t \\ 1 & \text{if} \quad x \;\geq\; 1+t \end{cases}$$

(The initial shock dissolves to an increasing linear function.)

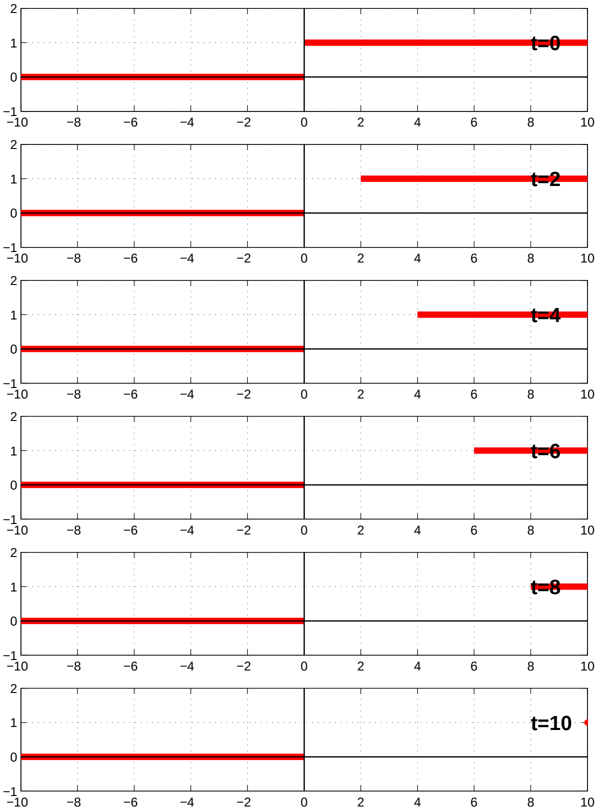

Figure 5:

5. If the initial condition is $\mathbf{U}_0 = 0\Xi_{]-\infty,0]} \cup x\Xi_{[0,1]} \cup \Xi_{[1,+\infty[}$, given by

$$\mathbf{U}_0(x) \ = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x \in [0,1] \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

then the solution $\mathbf{U}$ is given by

$$\mathbf{U}(t,x) \ = \ \Xi(]-\infty,0];x) \cup \left(\frac{x}{1+t}\right)\Xi([0,1+t];x) \cup \Xi([1+t,+\infty[;x) \ = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ \frac{x}{1+t} & \text{if} \quad 0 < x < 1+t \\ 1 & \text{if} \quad x \geq 1+t \end{cases}$$

It is single-valued as a function of x for all $t \geq 0$ (see Figure 6). The univocity of the viability solution is due to the increasing property of the initial value, a consequence of the general statement Proposition ??.

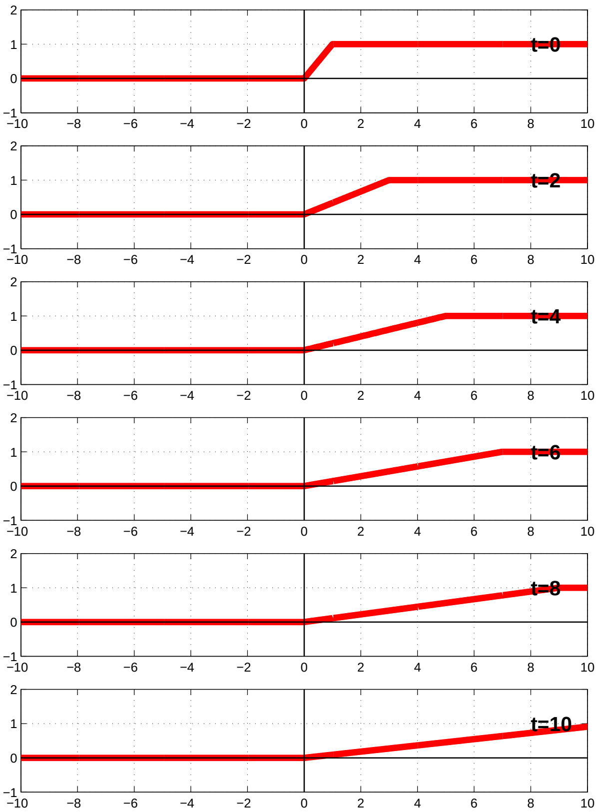

Figure 6:

6. If the initial condition is $\mathbf{U}_0 := \Xi_{]-\infty,0]} \cup (1-x)\Xi_{[0,1]} \cup 0\Xi_{[1,+\infty[}$ **is given by**

$$\mathbf{U}_0(x) \ = \begin{cases} 1 & \text{if} \quad x \leq 0 \\ 1-x & \text{if} \quad x \in [0,1] \\ 0 & \text{if} \quad x \geq 1 \end{cases}$$

Then one can check that the solution U is given by

$$\mathbf{U}(t,x) \ = \begin{cases} 0 & \text{if} \quad x < \min(t,1) \\ \frac{1-x}{1-t} & \text{if} \quad \min(t,1) \leq x < 1 \text{ (single-valued)} \\ [0,1] & \text{if} \quad x = 1 \ \& \ t = 1 \text{ (shock)} \\ \left\{0, 1, \frac{x-1}{t-1}\right\} & \text{if} \quad 1 < x \leq \max(t,1) \text{ (Z-shape)} \\ 1 & \text{if} \quad x \geq \max(t,1) \end{cases}$$

As a function of x**, the solution has a Z-shape after the shock that happened at** $t = 1$ **and** $x = 1$**. This is due to the fact that the initial condition** U_0 **is no longer increasing, and that it is single-valued on some finite time interval.**

Outline

1. *Introduction*
2. *Viability Solutions for Initial Value Problems*
3. *Morphism Property of the Viability Solution*
4. *Piecewise Linear Initial Conditions*
5. ► *Viability Solutions under Viability Constraints and/or Boundary Conditions*
6. *Derivatives of Set-Valued maps and Frankowska Solutions to the Burgers Equation*
7. *Regulating Controlled Burgers Tracking Problem*

Viability Constraints

Definition 6 ***Set*** $K := [\xi, +\infty[$ ***and the set-valued map*** Ψ_ξ ***defined by***

$$\Psi_\xi(t,x) \ := \begin{cases} [\underline{y}, \overline{y}] & \text{if} \quad x \ \geq \ \xi \\ \emptyset & \text{if} \quad x \ < \ \xi \end{cases}$$

We shall say that the set-valued map $\mathbf{U} : \mathbb{R}_+ \times \mathbb{R}_+ \rightsquigarrow \mathbb{R}$ ***defined by***

$$\text{Graph}(\mathbf{U}) \ := \ \text{Capt}_{(2)}(\text{Graph}(\Psi), \text{Graph}(\mathbf{U}_0^+)) \tag{5}$$

is the **viability solution** ***to the Burgers Tracking Problem (1) satisfying the initial condition*** $\mathbf{U}(0,x) := \mathbf{U}_0(x)$ ***under viability constraints*** $\mathbf{U}(t,x) \ \in \ [\underline{y}, \overline{y}]$***.***

Existence and Uniqueness

Then the viability solution defined by (5)

$$\text{Graph}(\mathbf{U}) \ := \ \text{Capt}_{(2)}(\text{Graph}(\Psi_\xi), \text{Graph}(\mathbf{U}_0^+))$$

is the unique solution to the tracking problem

$$\forall \, t \geq 0, \ \ y \ \in \ \mathbf{V}(t, x + (t-T)y) \cap [\underline{y}, \overline{y}] \cap \left[\underline{y}, \frac{x-\xi}{T}\right]$$

and the values $\mathrm{U}(T,x)$ are made of the fixed points

$$y \ \in \ \mathbf{U}_0(x - Ty) \cap [\underline{y}, \overline{y}] \cap \left[\underline{y}, \frac{x-\xi}{T}\right]$$

because $x - Ty$ must be larger than of equal to ξ. Such fixed points may no longer exist, and they never exist if $T > \dfrac{x-\xi}{\underline{y}}$.

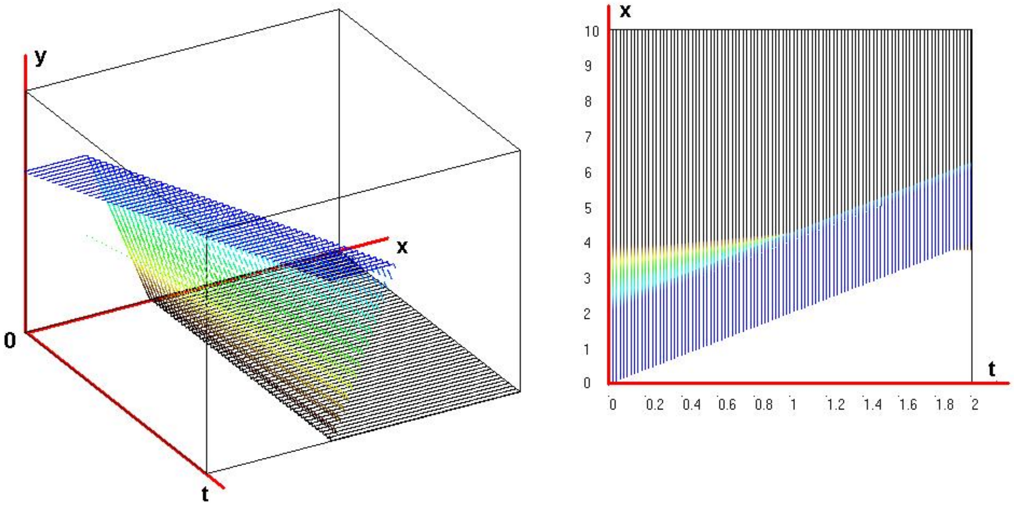

Figure 7: **Initial-Value Problem in the case of Constraints on Spatial Variables**. The viability solution is the capture basin of the graph of the initial condition $\mathbf{U}_0 := 2\Xi([0,1];x) \cup 2(2-x)\Xi([1,2];x) \cup 0\Xi([2,5];x)$. It has empty values if $T > \dfrac{x-\xi}{\underline{y}}$. Source: Patrick Saint-Pierre.

Boundary Conditions

For compensating such empty values, we introduce the sets $\Gamma_\xi(t,\xi)$ of velocities of elements arriving at time t at the entrance ξ of the road $K := [\xi, +\infty[$:

Definition 7 ***Assume that $\underline{y}$ is positive. Given the boundary condition $\Gamma_\xi : \mathbb{R}_+ \times K \rightsquigarrow Y$, a set-valued map $\mathbf{V} : \mathbb{R}_+ \times K \rightsquigarrow Y$ is a solution to the Burgers tracking problem if it satisfies***

$$y \in \mathbf{V}(T,x) \text{ if and only if } \forall\, t \geq T - \frac{x-\xi}{y} \geq 0, \;\; y \;\in\; \mathbf{V}\left(t, x+(t-T)y\right) \cap [\underline{y}, \overline{y}]$$

and the boundary condition

$$\forall\, t \geq 0 \;\; \mathbf{V}(t,\xi) \;=\; \Gamma_\xi(t,\xi)$$

Existence and Uniqueness

Theorem 8 *Assume that* $\underline{y} > 0$ *and that* $K := [\xi, +\infty]$*. The viability solution defined by*

$$\mathrm{Graph}(\mathbf{U}_\xi) \ := \ \mathrm{Capt}_{(2)}(\mathrm{Graph}(\Psi_\xi), \mathrm{Graph}(\Gamma_\xi)) \tag{6}$$

is the unique solution of the Burgers tracking problem (7) with boundary conditions.

Furthermore, $\mathrm{U}_\xi(T,x)$ *is the set of fixed point of the map*

$$y \rightsquigarrow \Gamma_\xi\left(T - \frac{x-\xi}{y}, \xi\right) \cap [\underline{y}, \overline{y}] \cap \left[\frac{x-\xi}{T}, \overline{y}\right]$$

where $T \geq \dfrac{x-\xi}{y}$ *(It is always empty when* $T < \dfrac{x-\xi}{\overline{y}}$*).*

It satisfies the "maximum principle"

$$\forall\, (t,x) \in \mathbb{R}_+ \times K, \quad \sup_{y \in \mathbf{U}_\xi(t,x)} |y| \ \leq \ \sup_{t \in \mathbb{R}_+} \ \sup_{y \in \Gamma_\xi(t) \cap [\underline{y}, \overline{y}]} |y|$$

or, more precisely

$$\forall\, (t,x) \in \mathbb{R}_+ \times K, \ \ \mathbf{U}_\xi(t,x) \ \subset \ \ \mathrm{Im}(\Gamma_\xi) \cap [\underline{y}, \overline{y}]$$

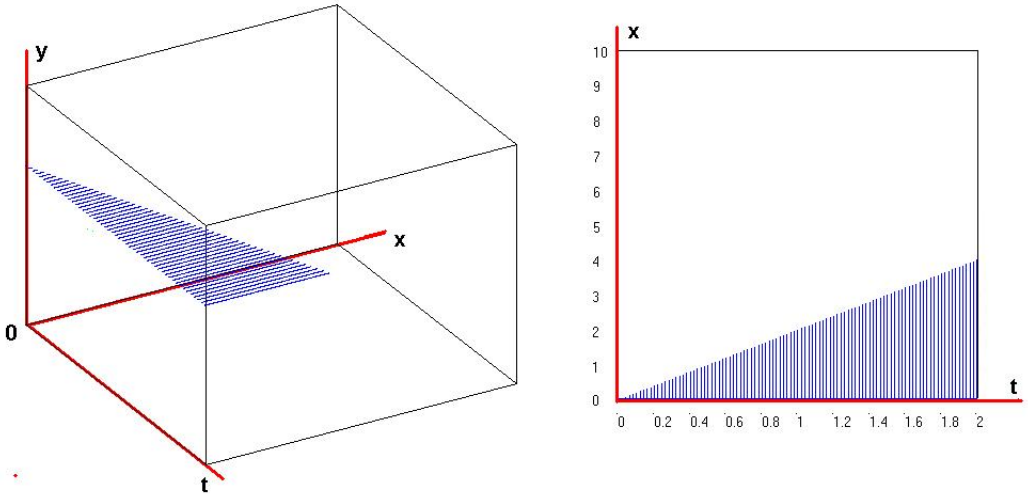

Figure 8: **Boundary-Value Problem in the case of Constraints on Spatial Variables**. The viability solution is the capture basin of the graph of the boundary condition $\Gamma_0(t) := 2\Xi([0,2];t)$. It has empty values if $T < \dfrac{x-\xi}{\overline{y}}$. Source: Patrick Saint-Pierre.

Characteristic Boundary Conditions

Proposition 9 ***The viability solution to the Burgers tracking problem (1) satisfying the boundary condition***

$$\Gamma_{\xi}(t) \; = \; \bigcup_{j \in \mathbb{J}} \delta_j \Xi_{\Delta_j}(t)$$

where $\Delta_j \subset \mathbb{R}_+$ ***are time intervals, is equal to:***

$$\mathbf{U}_{\xi}(t,x) \; = \; \bigcup_{j \in \mathbb{J}} \delta_j \Xi \left(\Delta_j + \frac{x-\xi}{\delta_j}; t \right)$$

which can be written in the form

$$\mathbf{U}(t,x) \; = \; \{\delta_j\}_{j \in \mathbb{J}(t,x)}$$

where $\mathbb{J}(t,x) \; := \; \left\{ j \in \mathbb{J} \;\; \text{such that}\; t \in \Delta_j + \frac{x-\xi}{\delta_j} \right\}$***, the cardinal of which plays the role of a*** ***“valuemeter”.***

Initial and Boundary Conditions

Theorem 10 ***Assume that*** $\underline{y} > 0$ ***and that*** $K := [\xi, +\infty]$ ***and that*** $\mathbf{U}_0(\xi) = \Gamma(0, \xi)$***. The union*** $(t, x) \rightsquigarrow \mathbf{U}(t, x) := \mathbf{U}_{\mathbf{U}_0}(t, x) \cup \mathbf{U}_\xi(t, x)$ ***of the viability solutions associated with the initial datum*** $\mathbf{U_0}$ ***and the boundary datum*** Γ_ξ ***is the unique solution of the Burgers tracking problem (7) with initial and boundary conditions***

$$\left\{ \begin{array}{lll} (i) & \forall\, x \geq \xi, & \mathbf{U}(0, x) \;=\; \mathbf{U}_0(x) \\ (ii) & \forall\, t \geq 0, & \mathbf{U}(t, \xi) \;=\; \Gamma_\xi(t, \xi) \end{array} \right.$$

Furthermore, $\mathrm{U}(T, x)$ ***is the set of velocities*** y ***satisfying***

$$\left\{ \begin{array}{lll} y \;\in\; \mathbf{U}_0(x - Ty) & \text{if} & T \;\leq\; \frac{x-\xi}{y} \\ y \;\in\; \Gamma_\xi\left(T - \frac{x-\xi}{y}, \xi\right) & \text{if} & T \;\geq\; \frac{x-\xi}{y} \end{array} \right.$$

It satisfies the "maximum principle"

$$\forall\, (t, x) \in \mathbb{R}_+ \times K, \;\; \mathbf{U}_\xi(t, x) \;\subset\; (\; \mathrm{Im}(\mathbf{U}_0) \cup \; \mathrm{Im}(\Gamma_\xi)) \cap [\underline{y}, \overline{y}]$$

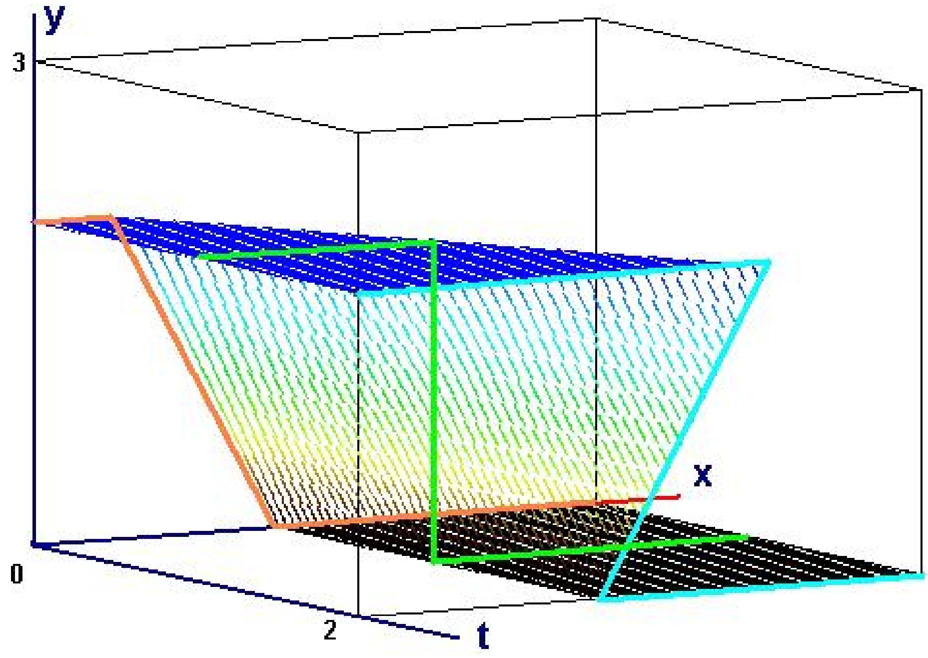

Figure 9: **Initial/Boundary-Value Problem in the case of Constraints on Spatial Variables**. The viability solution is the capture basin of the union of the graph of the initial condition $\mathbf{U}_0 := 2\Xi([0,1];x) \cup 2(2-x)\Xi([1,2];x) \cup 0\Xi([2,5];x)$ and of the graph of the boundary condition $\Gamma_0(t) := 2\Xi([0,2];t)$. See Figure 3. Source: Patrick Saint-Pierre.

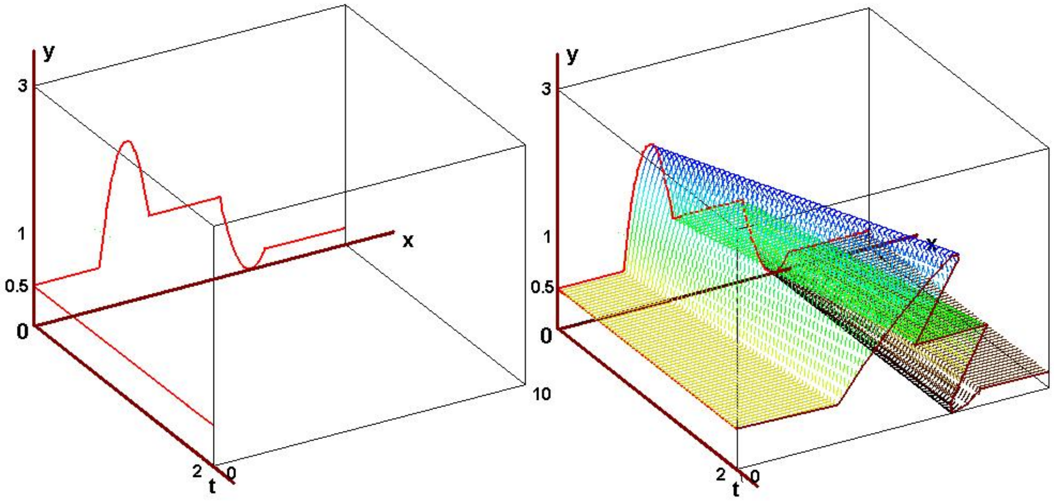

Figure 10: Example of viability solution to the Burgers' equation with initial data $\mathbf{U}_0(x) := \max(0.5, 2(1-(x-3)^2))\Xi([0,3];x) \cup \max(1, 2(1-(x-3)^2))\Xi([3,5];x) \cup \min(1,(x-7)^2)\Xi(5,7[];x) \cup \max(0.2,(x-7)^2)\Xi([7,10];x)$ and boundary condition $\Gamma_0(t) := 0.5\Xi([0,2];t)$.

Example of Non-Strict Boundary Condition

Figure 9 provided an example of a "well-posed" intial/boundary value-problem. We may consider an intermediate situation between this well-posed case and the case of an initial value problem without boundary condition as in Figure 7 by taking for boundary condition a non-strict set-valued map: See for instance Figure 11.

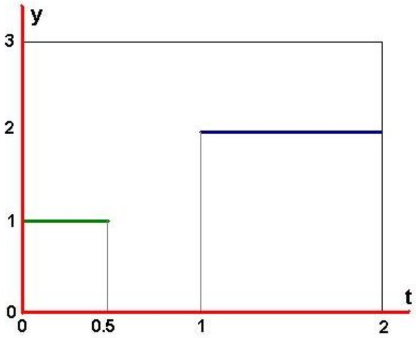

Figure 11: **Example of a Non-Strict Boundary Condition**, defined by the map $\Gamma_0(t) := \Xi([0, 0.5]; t) \cup 1.5\Xi([1, 2]; t)$ equal to 1 on $[0, 0.5]$, to 1.5 on $[1, 2]$ and to the empty set otherwise.

The viability solution of such a non-strict initial/boundary-value problem may have non-empty values: See Figure 12 for example.

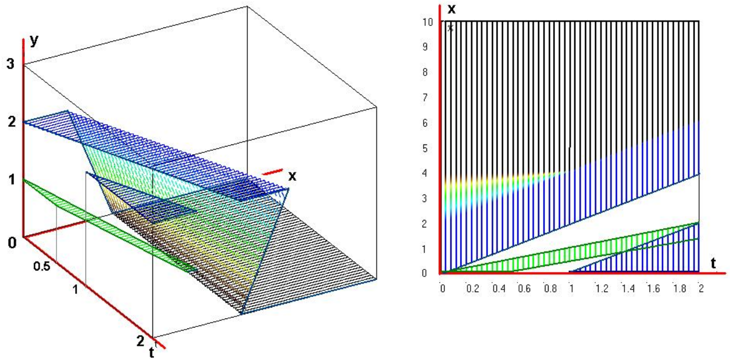

Figure 12: **Non-Strict Boundary-Value Problem in the case of constraints on spatial variables**. The viability solution is the capture basin of the union of graph of initial condition $\mathbf{U}_0 := 2\Xi([0,1];x) \cup 2(2-x)\Xi([1,2];x) \cup 0\Xi([2,5];x)$ and of the graph of the boundary condition defined by the map $\Gamma_0(t) := \Xi([0,0.5];t) \cup 1.5\Xi([1,2];t)$. It still has empty values in a subset of the area $T < \dfrac{x-\xi}{\overline{y}}$, but has also non empty values thanks to the contribution of the boundary condition, either equal to a singleton or to a subset of two elements. Source: Patrick Saint-Pierre.

Case of Several “Entrances”

Theorem 11 ***Assume that $\underline{y}$ is positive and let us consider crossroads $\xi =: \xi_0 < \xi_1 < \cdots < x_n$. Given the initial condition $\mathbf{U}_0$ and the entrance conditions Γ_{ξ_i}, $i = 0, 1, \ldots, n$, the union $\mathbf{U}$ of the viability solution $\mathbf{U}_{\mathbf{U}_0}$ associated with the initial condition $\mathbf{U}_0$ and of the viability solutions $\mathbf{U}_{\xi_i}$ associated with the entrance conditions Γ_{ξ_i} is the unique solution to the Burgers tracking problem satisfying: If $x \in [\xi_i, \xi_{i+1}[$, then $\mathrm{U}(T,x)$ is the set of velocities y such that:***

$$\begin{cases}
y \in \mathbf{U}_0(x-Ty) & \text{if } T \in \left[0, \dfrac{x-\xi_i}{y}\right[\\
y \in \mathbf{U}_0(x-Ty) \cup \Gamma_{\xi_i}\left(T-\dfrac{x-\xi_i}{y}, \xi_i\right) & \text{if } T \in \left[\dfrac{x-\xi_i}{y}, \dfrac{x-\xi_{i-1}}{y}\right[\\
\vdots & \vdots \quad \vdots \\
y \in \mathbf{U}_0(x-Ty) \cup \displaystyle\bigcup_{j=1}^{i} \Gamma_{\xi_j}\left(T-\dfrac{x-\xi_j}{y}, \xi_j\right) & \text{if } T \in \left[\dfrac{x-\xi_1}{y}, \dfrac{x-\xi_0}{y}\right[\\
y \in \displaystyle\bigcup_{j=0}^{i} \Gamma_{\xi_j}\left(T-\dfrac{x-\xi_j}{y}, \xi_j\right) & \text{if } T \geq \dfrac{x-\xi_0}{y}
\end{cases}$$

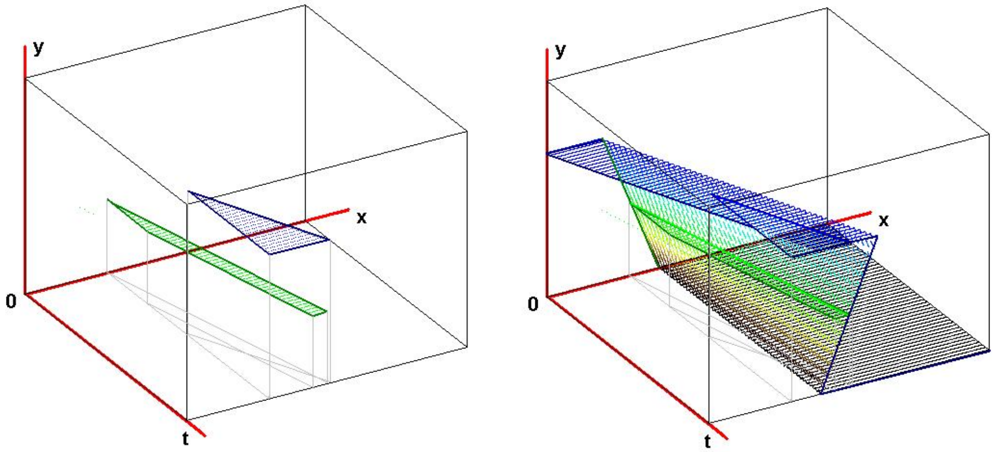

Figure 13: **Example of an "Entrance" Condition** $\Gamma_1(t) := \Xi([0, 0.5]; t) \cup 1.5\Xi([1, 2]; t)$ alone on the left and with initial condition $\mathbf{U}_0 := 2\Xi([0, 1]; x) \cup 2(2 - x)\Xi([1, 2]; x) \cup 0\Xi([2, 5]; x)$ on the right.

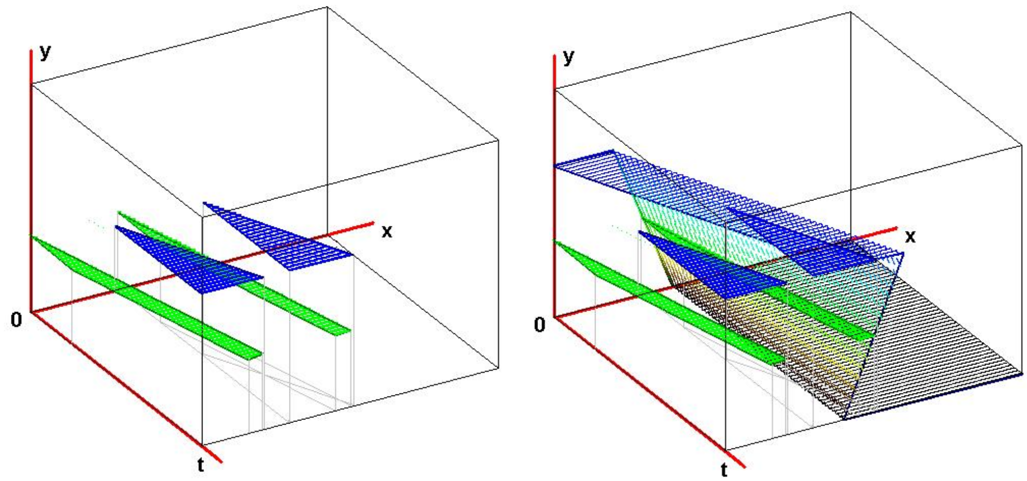

Figure 14: **Example of multiple “Entrance” Conditions** as the union of an boundary condition $\Gamma_0(t) := \Xi([0,0.5];t) \cup 1.5\Xi([1,2];t)$ and of the entrance condition $\Gamma_1(t) := \Xi([0,0.5];t) \cup 1.5\Xi([1,2];t)$ alone on the left and with initial condition $\mathbf{U}_0 := 2\Xi([0,1];x) \cup 2(2-x)\Xi([1,2];x) \cup 0\Xi([2,5];x)$ on the right.

Time-Dependent Constraints

A natural viability condition is that for each $t \geq 0$, $x(t) := x + ty \leq a(t)$ where $a(\cdot) \geq \xi$ is a time function.

This can be taken into account by introducing the a new constrained map Ψ^a defined by

$$\Psi^a(t,x) \ := \left\{ \begin{array}{lll} [\underline{y}, \overline{y}] & \text{if} & x \ \in \ [\xi, a(t)] \\ \emptyset & \text{otherwise} & \end{array} \right.$$

Therefore, we associate with the initial datum U_0 its restriction U_0^a defined by

$$\mathbf{U}_0^a(t,x) \ := \left\{ \begin{array}{lll} \mathbf{U}_0(x) & \text{if} & t = 0 \ \& \ x \ \in \ [\xi, a(0)] \\ \emptyset & \text{otherwise} & \end{array} \right.$$

Its viability solution is defined by

$$\text{Graph}(\mathbf{U}^a) \ := \ \text{Capt}_{(2)}(\text{Graph}(\Psi^a), \text{Graph}(\mathbf{U}_0^a \cup \Gamma_\xi)) \qquad (7)$$

and can be computed by the Saint-Pierre Capture Basin Algorithm.

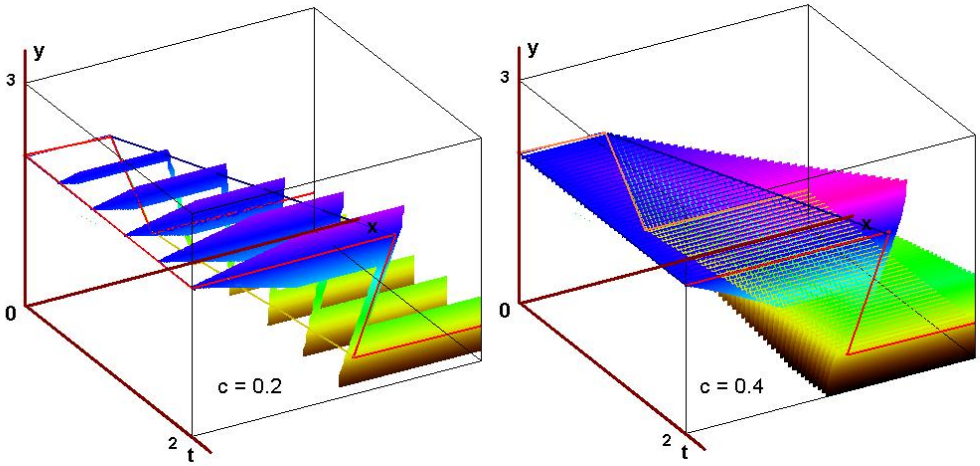

Figure 15: **Viability Solution to the Controlled Burgers' Equation** $\frac{\partial \mathbf{U}(t,x)}{\partial t} + \frac{\partial \mathbf{U}(t,x)}{\partial x}\mathbf{U}(t,x) = u$ where $u \in [-c,+c]$ for 2 values of $c = 0.2$ & 0.4 with initial condition equal to $\mathbf{U}_0 := 2\Xi([0,1];x) \cup 2(2-x)\Xi([1,2];x) \cup 0\Xi([2,5];x)$ and boundary condition equal to $\Gamma_0(t) := \Xi([0,0.5];t) \cup 1.5\Xi([1,2];t)$. It still has the familiar Z-shape, but with "thick" values. They are "separated" for $c = 0.2$ for the sake of clarity.

Outline

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4. *Piecewise Linear Initial Conditions*
5. *Viability Solutions under Viability Constraints and/or Boundary Conditions Regulating Controlled Burgers Tracking Problem*
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Contingent Cones

Graphical derivatives of set-valued maps were defined in 1981.

We need first introduce the contingent cone to a subset K at a point $x \in K$, introduced in the early thirties independently by Bouligand and Severi, for adapting to any subset the concept of tangent space to manifolds: A direction $v \in X$ belongs to $T_K(x)$ if there exist sequences $h_n > 0$ and $v_n \in X$ converging to 0 and v respectively such that

$$\forall\, n \geq 0,\;\; x + h_n v_n \;\in\; K$$

This means that the contingent cone is the Painlevé-Kuratowski upper limit of the subsets $\frac{K-x}{h}$ when h converges to 0.

Derivatives of Set-Valued Maps

The graph of the set-valued map $DV(t,x,y)$ from $\mathbf{R}\times X$ to Y is equal to the contingent cone to the graph of V at (t,x,y):

$$T_{\mathrm{Graph}(V)}(t,x,y) \ = \ \mathrm{Graph}(DV(t,x,y))$$

This is how Pierre de Fermat defined in 1637 the derivative of a function as the slope of the tangent to its graph.

Consequently, to say that $g \in Y$ belongs to the contingent derivative $DV(t,x,y)(\pm 1,f)$ of V at (t,x,y) in the direction $(\pm 1,f) \in \mathbf{R}\times X$ means that

$$\liminf_{h\to 0+,\ f'\to f} \left\| \frac{V(t\pm h, x+hf')-y}{h} - g \right\| \ = \ 0$$

Examples

When $u : \mathbf{R} \times X \mapsto Y$ is single-valued, we set $Du(t,x) := Du(t,x,u(t,x))$.

$Du(t,x)(\pm 1, f) = \pm \frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} \cdot f$ **whenever u is differentiable at (t,x).**

When u is Lipschitz on a neighborhood of (t,x) and when the dimension of X is finite, the domain of $Du(t,x)$ is not empty.

Furthermore, the Rademacher Theorem implies that $x \rightsquigarrow Du(t,x)$ is almost everywhere single-valued. Equality $Du(t,x)(-1,-f) = -Du(t,x)(1,f)$ is not true in general.

Frankowska solution to the Burgers equation

Being a capture basin, the graph of the viability solution is characterized by tangential conditions that imply that the viability solution to the Burgers equation is actually the unique solution (with closed graph) in the Frankowska sense to the Burgers equation, when the derivatives are meant to be the contingent derivatives of a set-valued map:

Theorem 12 ***The viability solution*** $\mathbf{U}$ ***is the unique Frankowska solution to the Burgers equation (1) satisfying the initial condition*** $\mathbf{U}(0,x) := \mathbf{U}_0(x)$ ***in the sense that***

$$\left\{ \begin{array}{l} \forall\, t > 0,\ \forall\, x \in \mathbb{R}, \forall\, y \in \mathbf{U}(t,x),\ \ 0\ \in\ D\mathbf{U}(t,x,y)(-1,-y) \\ \forall\, t \geq 0,\ \forall\, x \in \mathbb{R}, \forall\, y \in \mathbf{U}(t,x),\ \ 0\ \in\ D\mathbf{U}(t,x,y)(1,y) \end{array} \right.$$

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General Case

Let us consider

1. **two finite dimensional vector spaces X and Y, a closed subset $K \subset X$,**
2. **two maps $\mathbf{a} : \mathbb{R}_+ \times X \times Y \times \mathcal{P} \mapsto X$ and $f : \mathbb{R}_+ \times X \times Y \times \mathcal{P} \mapsto Y$**
3. **A set-valued map $P : \mathbb{R}_+ \times X \times Y \mapsto \mathcal{P}$,**
4. **a set-valued map $\Psi : \mathbb{R}_+ \times X \mapsto Y$ defining the viability constraints (which may be time and state dependent),**
5. **a set-valued map $\Phi : \mathbb{R}_+ \times X \mapsto Y$ contained in Ψ defining the initial conditions.**

Definition 13 ***We shall say that a set-valued map $\mathbf{V} : \mathbb{R}_+ \times X \rightsquigarrow Y$ between Φ and Ψ is a solution to the tracking problem under the control system***

$$\left\{\begin{array}{lrcl} (i) & x'(t) & = & \mathbf{a}(t, x(t), y(t), u(t)) \\ (ii) & y'(t) & = & g(t, x(t), y(t), u(t)) \\ & \text{where } u(t) & \in & P(t, x(t), y(t)) \end{array}\right. \tag{8}$$

if for every $y \in \mathbf{V}(T, x)$

1. ***there exist at least one solution*** $(x(\cdot), y(\cdot))$ ***to the control system (8) and*** $s^\star \in [0, T]$ ***satisfying***

$$\begin{cases} (i) & x(T) = x \text{ and } y(T) = y \\ (ii) & y(s^\star) \ \in \ \Phi(s^\star, x(s^\star)) \\ (iii) & \forall\, t \in [s^\star, T], \quad y(t) \in \mathbf{V}(t, x(t)) \end{cases}$$

2. ***all solutions*** $(x(\cdot), y(\cdot))$ ***to control system (8) such that***

$$\begin{cases} (i) & x(T) = x \text{ and } y(T) = y \\ (ii) & \text{for all } t \in [T, s],\ y(t) \in \Psi(t, x(t)) \end{cases}$$

   satisfy

$$\text{for all } t \in [T, s],\ y(t) \in \mathbf{V}(t, x(t))$$

Viability Solution to the Tracking Problem

Definition 14 ‘ ***Let us introduce the “characteristic control system”***

$$\left\{\begin{array}{llll} (i) & \tau'(t) & = & -1 \\ (ii) & x'(t) & = & -\mathbf{a}(\tau(t), x(t), y(t), u(t)) \\ (iii) & y'(t) & = & -g(\tau(t), x(t), y(t), u(t)) \\ & \text{where } u(t) & \in & P(\tau(t), x(t), y(t)) \end{array}\right. \quad (9)$$

We shall say that the set-valued map $\mathbf{U}_{(\Psi,\Phi)} : \mathbb{R}_+ \times K \rightsquigarrow Y$ ***defined by***

$$\text{Graph}(\mathbf{U}_{(\Psi,\Phi)}) := \text{Capt}_{(9)}(\text{Graph}(\Psi), \text{Graph}(\Phi)) \quad (10)$$

is the **viability solution** ***to the tracking problem. When there is no ambiguity, we set*** $\mathbf{U} := \mathbf{U}_{(\Psi,\Phi)}$***.***

Existence and Uniqueness of the Solution to the Tracking Problem

Theorem 15 ***Assume that the maps*** a,g ***and*** P ***are Marchaud and that the set-valued maps*** Φ ***and*** Ψ ***are closed.***

Then the viability solution $\mathrm{U}_{(\Phi,\Psi)}$ ***is the*** unique ***solution*** V ***between*** Φ ***and*** Ψ ***to the tracking problem.***

A Formula for the Viability Solution

Theorem 16 ***The value*** $\mathrm{U}(T,x)$ ***of the viability solution is the set of fixed points*** $y \in Y$ ***of***

$$\left\{\begin{array}{l} y \;\in\; \Phi(T-\overleftarrow{\tau}^{\sharp}(T,x,y), x^{(T,x,y)}(\overleftarrow{\tau}^{\sharp}(T,x,y))) + \int_0^{\overleftarrow{\tau}^{\sharp}(T,x,y)} g(T-\tau, x^{(T,x,y)}(\tau), y^{(T,x,y)}(\tau))d\tau \\ \cap \bigcap_{t\in[0,\overleftarrow{\tau}^{\sharp}(T,x,y)]} \left(\Psi(T-t, x^{(T,x,y)}(t)) + \int_0^t g(T-\tau, x^{(T,x,y)}(\tau), y^{(T,x,y)}(\tau), u(\tau))d\tau\right) \end{array}\right.$$

where $(x^{(T,x,y)}(\cdot), y^{(T,x,y)}(\cdot))$ ***denotes a solution to the backward backward controlled system***

$$\left\{\begin{array}{lrcl} (i) & x'(t) & = & -\mathbf{a}(T-t, x(t), y(t), u(t)) \\ (ii) & y'(t) & = & -g(T-t, x(t), y(t), u(t)) \end{array}\right.$$

starting at (x,y) ***at initial time until the time*** $\overleftarrow{\tau}^{\sharp}(T,x,y) \leq +\infty$ ***denoting the first instant*** t ***when the evolution*** $y^{(T,x,y)}(t)$ ***leaves*** $\mathbf{U}(T-t, x^{(T,x,y)}(t))$***.***

If the graph of Ψ ***is invariant, then*** $\mathbf{U}(T,x)$ ***is the set of fixed points*** $y \in Y$ ***of***

$$y \;\in\; \Phi(T-\overleftarrow{\tau}^{\sharp}(T,x,y), x(\overleftarrow{\tau}^{\sharp}(T,x,y))) + \int_0^{\overleftarrow{\tau}^{\sharp}(T,x,y)} g(T-\tau, x^{(T,x,y)}(\tau), y^{(T,x,y)}(\tau), u(\tau))d\tau$$

Contingent and Frankowska Solutions

Definition 17 *We shall say that a set-valued map* $\mathbf{V} : \mathbb{R}_+ \times X \mapsto Y$ *between* Φ *and* Φ **with closed graph** *is a* **contingent solution to the semi-linear system of first-order partial differential inclusions**

$$\begin{cases} (i) & \forall\, y \in \mathbf{V}(t,x),\ \ \dfrac{\partial \mathbf{V}(t,x)}{\partial t} + \dfrac{\partial \mathbf{V}(t,x)}{\partial x}\mathbf{a}(t,x,\mathbf{V}(t,x),u) \ = \ f(t,x,\mathbf{V}(t,x),u) \\ (ii) & \text{where } u \in P(t,x,y) \end{cases} \tag{11}$$

if it satisfies

$$\begin{cases} \forall\, t > 0,\ \forall\, x \in \mathrm{Int}(K), \forall\, y \in \mathbf{V}(t,x),\ \exists\, u \in P(t,x,y) \\ \ \ \textit{such that}\ \ 0 \ \in \ D\mathbf{V}(t,x,y)(-1,-\mathbf{a}(t,x,y,u)) + f(t,x,y,u) \end{cases} \tag{12}$$

A set-valued map $\mathbf{V}$ *is said to be a* **Frankowska solution** *if it satisfies (12) and the Frankowska property*

$$\begin{cases} \forall\, t \geq 0,\ \forall\, x \in K, \forall\, y \in \mathbf{V}(t,x),\ \forall\, u \in P(t,x,y), \\ 0 \ \in \ D\mathbf{V}(t,x,y)(+1,+\mathbf{a}(t,x,y,u)) - f(t,x,y,u) \end{cases} \tag{13}$$

The Main Theorem

Theorem 18 *Assume that the maps* a*,* f *and* P *are Marchaud and that the set-valued maps* $\mathbf{U}_0$*,* Γ_K *and* Ψ *are closed.*

1. *Then the viability solution* $\mathbf{U}$ *is the largest contingent solution to (11) between* Φ *and* Ψ*.*
2. *If we assume further that the maps* a*,* f *and* P *are Lipschitz and if there is no viability constraint, then the viability solution is the* ***unique Frankowska solution*** *to ((12),(13)).*

The Regulation Theorem for the Tracking Problem

Theorem 19 ***Assume that the maps*** $\mathbf{a}$***,*** f ***and*** P ***are Marchaud and that the set-valued maps*** $\mathbf{U}_0$***,*** Γ_K ***and*** Ψ ***are closed.***

Let us set

$$R(t,x,y) \ := \ \{u \in P(t,x,y) \mid 0 \ \in \ D\mathbf{V}(t,x,y)(-1,-\mathbf{a}(t,x,y)) + f(t,x,y)\}$$

For any $y \in \mathbf{U}(T,x)$***, the evolutions satisfying*** $y(t) \in \mathbf{U}(t,x(t))$***,*** $x(T) = x$ ***and*** $y(T) = y$ ***are governed by the controlled system***

$$\left\{ \begin{array}{lll} (i) & x'(t) & = \ \mathbf{a}(t,x(t),y(t),u(t)) \\ (ii) & y'(t) & = \ g(t,x(t),y(t),u(t)) \\ & \text{where } u(t) & \in \ R(T-t,x(t),y(t)) \end{array} \right. \tag{14}$$

until some finite time $s^\star$ ***where***

$$y(s^\star) \ \in \ \Phi(s^\star, x(s^\star))$$

A Priori Estimates

Theorem 20 *Assume that the control system (8)*

$$\left\{\begin{array}{lrcl} (i) & x'(t) & = & \mathbf{a}(t,x(t),y(t),u(t)) \\ (ii) & y'(t) & = & g(t,x(t),y(t),u(t)) \\ & \text{where } u(t) & \in & P(t,x(t),y(t)) \end{array}\right.$$

is Marchaud map satisfies growth conditions:

$$\left\{\begin{array}{ll} (i) & \sup\limits_{(t,x,y)\in \mathrm{Graph}(\Psi)} \sup\limits_{u\in P(t,x,y)} \|\mathbf{a}(t,x,y,u)\| \ \leq \ c(\|x\|+1) \\ (ii) & \sup\limits_{(t,x,y)\in \mathrm{Graph}(\Psi)} \sup\limits_{u\in P(t,x,y)} \dfrac{\langle g(t,x,y,u),y\rangle}{\|y\|} \ \leq \ -\lambda\|y\| + d(\|x\|+1)^{\alpha} \end{array}\right.$$

and that the growth of U_0 ***is polynomial:***

$$\forall\, x \in \mathrm{Dom}(\mathbf{U}_0), \quad \|\mathbf{U}_0(x)\| \ \leq \ b(\|x\|+1)^{\beta}$$

Let $\mathbf{U}_{(\Psi,\mathbf{U}_0)} : \mathbb{R}_+ \times X \rightsquigarrow Y$ ***be the viability solution to the tracking problem.***

1. ***if*** $\lambda = c\alpha$***,***

$$\|\mathbf{U}_{(\Psi,\mathbf{U}_0)}(T,x)\| \;\leq\; be^{c(\beta-\alpha)T}(\|x\|+1)^{\beta} + Td(\|x\|+1)^{\alpha}$$

2. ***if*** $\lambda > c\alpha$***,***

$$\|\mathbf{U}_{(\Psi,\mathbf{U}_0)}(T,x)\| \;\leq\; be^{-(\lambda-c\beta)T}(\|x\|+1)^{\beta} + \frac{d(\|x\|+1)^{\alpha}}{\lambda-c\alpha}\left(1-e^{-(\lambda-c\alpha)T}\right)$$

Consequently, if $\lambda > c\max(\alpha,\beta)$***, then***

$$\|\mathbf{U}_{(\Psi,\mathbf{U}_0)}(T,x)\| \;\leq\; b(\|x\|+1)^{\beta} + \frac{d(\|x\|+1)^{\alpha}}{\lambda-c\alpha}$$

Asymptotic properties

Let us consider an initial datum $\mathbf{U}_0 : X \leadsto Y$ and the viability solution $\mathbf{U}_{(\Psi,\mathbf{U}_0)}$ under the time-dependent tracking problem under control problem

$$\left\{\begin{array}{lrcl} (i) & x'(t) & = & \mathbf{a}(x(t),y(t),u(t)) \\ (ii) & y'(t) & = & g(x(t),y(t),u(t)) \\ & \text{where } u(t) & \in & P(x(t),y(t)) \end{array}\right. \tag{15}$$

with initial condition $\mathbf{U}_{(\Psi,\mathbf{U}_0)}(0,x) = \mathbf{U}_0(x)$, the graph of which is defined by

$$\text{Graph}(\mathbf{U}_{(\Psi,\mathbf{U}_0)} \ := \ \text{Capt}_{(17)}(\text{Graph}(\Psi),\text{Graph}(\mathbf{U}_0)) \tag{16}$$

where

$$\left\{\begin{array}{lrcl} (i) & \tau'(t) & = & -1 \\ (ii) & x'(t) & = & -\mathbf{a}(x(t),y(t),u(t)) \\ (iii) & y'(t) & = & -g(x(t),y(t),u(t)) \\ & \text{where } u(t) & \in & P(x(t),y(t)) \end{array}\right. \tag{17}$$

The Convergence Theorem

The question arises to know whether the "limit" of the solutions $\mathbf{U}_{(\Psi,\mathbf{U}_0)}(T,\cdot)$ when $T \rightarrow +\infty$ is a solution to the time-independent tracking problem.

Theorem 21 ***Let us consider the graphical upper limit*** $\mathbf{U}^{\sharp}_{(\Psi,\mathbf{U}_0)} : X \rightsquigarrow Y$ ***when*** $T \rightarrow +\infty$ ***of the viability solution*** $\mathbf{U}_{(\Psi,\mathbf{U}_0)}(T) : X \rightsquigarrow Y$ ***defined by*** $\mathbf{U}_{(\Psi,\mathbf{U}_0)}(T)(x) := \mathbf{U}_{(\Psi,\mathbf{U}_0)}(T,x)$***:***

$$\mathrm{Graph}(U^{\sharp}_{(\Psi,\mathbf{U}_0)}) \;:=\; \mathrm{Limsup}_{T\rightarrow+\infty}\mathrm{Graph}(U_{(\Psi,\mathbf{U}_0)}(T))$$

Assume that controlled system is both Marchaud and Lipschitz. Then $U^{\sharp}_{(\Psi,\mathbf{U}_0)}$ ***is a subsolution to the tracking problem in the sense that it satisfies:***

For every $(x,y) \in \text{Graph}(U^{\sharp}_{(\Psi,\mathbf{U}_0)})$***,***

1. ***there exists a solution*** $(x(\cdot),y(\cdot))$ ***to the control system (15) arriving at*** (x,y) ***at time*** 0 ***and satisfying***

$$\forall\, t \leq 0, \quad y(t) \;\in\; U^{\sharp}_{(\Psi,\mathbf{U}_0)}(x(t))$$

2. ***all solutions*** $(x(\cdot),y(\cdot))$ ***to control system (15) starting at*** (x,y) ***and satisfying***

$$\text{for all } t \in [0,s], \quad y(t) \in \Psi(x(t))$$

satisfy

$$\text{for all } t \in [0,s],\ y(t) \in \mathbf{U}^{\sharp}_{(\Psi,\mathbf{U}_0)}(x(t))$$

Nonlinear Hadamard Problem

We may know that there exists a viability solution to a time-independent tracking problem, but nothing guarantees that the values U_Ψ are not empty.

Even for linear problems, we know since Jacques Hadamard the viability solution to the tracking problem under a system

$$\left\{\begin{array}{lrcl} i) & x'(t) & = & \mathbf{a}(x(t)) \\ ii) & y'(t) & = & -My(t)+g(x(t)) \end{array}\right.$$

has a solution equal to

$$\mathbf{U}(x) \;:=\; -\int_0^{+\infty} e^{-Mt} g(\vartheta_{\mathbf{a}}(t,x))dt$$

it the largest eigenvalue λ is large enough for the above integral to exist.

The question arises whether the viability solution $\mathrm{U} : X \rightsquigarrow Y$ to the tracking problem is strict, i.e., that it has nonempty values: We present a theorem due to Hélène Frankowska extending Hadamard's Theorem to nonlinear control systems subjected to adequate growth conditions.

Frankowska Non-Emptiness Theorem

Theorem 22 (Frankowska) *Assume that the dynamics of the controlled system (17) are are Marchaud maps satisfying growth conditions:*

$$\begin{cases} (i) & \sup_{(\|y\|\leq\frac{d}{\lambda-c\alpha}(\|x\|+1)^{\alpha}}\ \sup_{u\in P(x,y)} \|\mathbf{a}(x,y,u)\| \ \leq \ c(\|x\|+1) \\ (ii) & \sup_{\|y\|\leq\frac{d}{\lambda-c\alpha}(\|x\|+1)^{\alpha}}\ \sup_{u\in P(x,y)} \dfrac{\langle g(x,y,u),y\rangle}{\|y\|} \ \leq \ -\lambda\|y\|+d(\|x\|+1)^{\alpha} \end{cases}$$

If $\lambda > c\alpha$, then there exists a solution $\mathbf{U} : X \rightsquigarrow Y$ with nonempty values to the tracking problem satisfying the growth condition

$$\forall\, x \in X,\ \ \|\mathbf{U}(x)\| \ \leq \ \frac{d}{\lambda - c\alpha}(\|x\|+1)^{\alpha}$$

which is both backward viable and invariant under (15).

It is the largest contingent solution $\mathbf{U} : X \rightsquigarrow Y$ satisfying the above growth condition to the partial differential inclusion

$$\forall\, x, \forall\, y \in \mathbf{U},\ \exists\, u \in P(x,y) \mid 0 \ \in \ \frac{\partial \mathbf{U}}{\partial x}\mathbf{a}(x,y,u) - g(x,y,u)$$

Merci pour votre
Thanks for your
Attention


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