ADVANCED METHODS FOR DECISION MAKING AND RISK MANAGEMENT IN SUSTAINABILITY SCIENCE

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EDITORS

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Chapter 2

AN INTRODUCTION TO VIABILITY THEORY AND MANAGEMENT OF RENEWABLE RESOURCES

Jean-Pierre Aubin and Patrick Saint-Pierre

2.1. Introduction

The main purpose of viability theory is to explain the evolution of the state of a control system. A control system is governed by non-deterministic dynamics and subjected to viability constraints that reveal concealed feedbacks which allow the system to be regulated and provide selection mechanisms for implementation. It assumes implicitly an "opportunistic" and "conservative" behavior of the system. In other words a behavior which enables the system to keep viable solutions, as long as its potential for exploration (or its lack of determinism), described by the availability of several evolutions, makes possible its regulation.

We illustrate the main concepts and results of viability theory by revisiting the Verhulst type models in population dynamics and adapting these models to the management of renewable resources. The class of all Malthusian feedbacks (mapping states to growth rates) guarantee the viability of the evolutions.

Other examples of viability constraints are provided by architectures of networks described by connectionist tensors operating on coalitions of actors linked by the network. It raises the question how to modify a given dynamical system governing the evolution of the signals, the connectionist tensors and the coalitions, in such a way that the architecture remains viable.

2.1.1. From Malthus to Verhulst and Beyond

A population for which there is a constant supply of resources and no predators provides a simple one-dimensional example: at each instant $t \geq 0$, the population $x(t)$ must remain confined in an interval $K := [a,b]$ where $0 < a < b$. The maximal population size b is called the *carrying capacity*.

The dynamics are unknown, and several models have been proposed. They are all particular cases of a general dynamical systems of the form

$$x'(t) \;=\; \widetilde{u}\big(x(t)\big)\; x(t)\,,$$

where $\widetilde{u} : [a,b] \mapsto \mathbb{R}$ is a model of the growth rate of the population, feeding back on the size of the population. In 1798, Malthus advocated in 1798 to choose a constant positive growth rate $\widetilde{u}(x) = u > 0$, leading to an exponential evolution $x(t) = xe^{ut}$ starting at x, which cannot be viable in any bounded interval. This is the price to pay for linearity of the dynamic of the population: "*population, when unchecked, increases in a geometrical ratio*".

Population dynamical models provide fast growing evolutions when the population is small and a slower rate when it becomes large. The *purely logistic Verhulst model* compensates for the never ending expansion of the Mathusian model, with feedback of the form

$$\widetilde{u}(x) \;:=\; r(b-x)\,,$$

proposed in 1838 by the Belgian mathematician Pierre-François Verhulst (rediscovered in the 1930's by Raymond Pearl). The solution to the logistic differential equation $x'(t) = rx(t)\big(b - x(t)\big)$ starting from $x \in [a,b]$ are respectively equal to

$$x(t) \;=\; \frac{bx}{x + (b-x)e^{-rt}}\;.$$

They remain confined in the interval $[a,b]$ and converge to the equilibrium b when $t \mapsto +\infty$. The logistic model and the S-shape graph of its solution became popular in the 1920's and stood as the evolutionary model of a large manifold of growths, from the tail of rats to the size of men.

Instead of finding one feedback $\widetilde{u}$ to satisfy the above requirements by trial and error, we proceed systematically to design feedbacks by leaving the choice of the growth rates open, regarding them as regulons of the system

$$x'(t) \;=\; u(t)\,x(t)\,. \tag{2.1}$$

These regulons are chosen to govern evolutions confined in the interval $[a,b]$. Imposing a bound on the velocities of the population's growth rates further restrict the selection of such regulons.

We suggest to characterize all of the feedbacks governing evolutions viable on the interval $[a,b]$ under inertia bound c. Among them, we will find the Verhulst feedback, to which we add two explicit other ones, providing inert and heavy evolutions. Inert evolutions are obtained by taking the maximal velocities allowed: $u'(t) = \pm c$.

Heavy evolutions combine Malthusian and inert growth. They are obtained by starting with the constant growth rate (Malthusian evolution) of the state (by taking $u'(t) = 0$) until the time when the inertia bound is met. This provides *warning signals with when, where and how the regulons must evolve*: the feedback becomes a specific inert feedback, providing constant (negative) velocities $u'(t) = -c$ of growth rates driving the state at its carrying capacity b.

Before detailing these facts and describing other results, we need to describe the concept of viability kernel, a central concept of viability theory.

2.1.2. Purpose of this Chapter

The purpose of this paper is to present a brief introduction of the viability theory that studies adaptive type evolution of complex systems under uncertain environments and viability constraints, found in many domains ranging from living beings to biology and cognitive sciences, ecology, sociology and economics.

Instead of applying only known mathematical and algorithmic techniques, most apply physics and are not necessarily adapted to such problems. Viability theory both designs and develops mathematical and algorithmic methods for studying the evolution of such systems, organizations and networks of systems that are:

1. constrained to adapt to a (possibly co-evolving) environment
2. evolving under contingent, stochastic or tychastic[1] uncertainty,
3. using regulation controls and in the case of networks, connectionist matrices or tensors
4. the evolution of which is governed by regulation laws that are then "computed", according to given principles such as the inertia principle
5. the evolution being either continuous, discrete, or an "hybrid" of the two when impulses are involved
6. the evolution concerning both the variables and the environmental constraints mutational viability),
7. the non-viable dynamics being corrected by introducing adequate controls when necessary (viability multipliers)
8. by introducing the "viability kernel" of a constrained set under a nonlinear controlled system (either continuous or hybrid), that is the set of initial states from which starts at least one evolution reaching a target in finite time while obeying state (viability) constraints.

It is a consensus that the evolution of many variables that describe systems, organizations and networks arising in biology, human, and social sciences, do not evolve in a deterministic way, and perhaps, not even in a stochastic way as it is usually understood, but with a Darwinian flavor, where inter-temporal optimality selection mechanisms are replaced by several forms of "viability", a word encompassing polysemous concepts such

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[1] Uncertainty is translated mathematically by parameters on which actors, agents, decision makers, etc. These parameters are often perturbations, disturbances (as in "robust control" or "differential games against nature") or more generally, tyches (meaning "chance" in classical Greek, from the Goddess Tyche) ranging over a state-dependent tychastic map. They could be called "random variables" if this vocabulary were not already confiscated by probabilists. This is why we borrow the term of *tychastic evolution* to Charles Peirce who introduced it in a paper published in 1893 under the title *evolutionary love*: *"Three modes of evolution have thus been brought before us: evolution by fortuitous variation, evolution by mechanical necessity, and evolution by creative love. We may term them tychastic evolution, or tychasm, anancastic evolution, or anancasm, and agapastic evolution, or agapasm.* One can prove that stochastic viability is a (very) particular case of tychastic viability (see Aubin and Da Prato 1998 and Aubin and Doss 2003 for instance).

as stability, confinement, and homeostasis, in which (see, e.g., Petschel-Held et al. 1999; Bruckner et al. 1999; Tóth 2003) the idea is expressed that some variables must obey some constraints. Inter-temporal optimization is replaced by myopic selection mechanisms that involve present knowledge, sometimes the knowledge of the history (or the path) of the evolution, instead of anticipations or knowledge of the future (whenever the evolution of these systems cannot be reproduced experimentally). Uncertainty does not necessarily obey statistical laws, but only unpredictable rare events (tyches, or perturbations, disturbances) that obey no statistical law and that must be avoided at all costs (precautionary principle or robust control). These systems can be regulated by using regulation (or cybernetical) controls that have to be chosen as feedbacks for guaranteeing the viability of a system and/or the capturability of targets and objectives, possibly against tyches (perturbations played by nature). The sets of controls as well as the constrained set can be "toll sets" (a variant of *fuzzy sets*) as in Aubin and Dordan 1996, for instance.

The purpose of viability theory is to attempt to answer directly the question that some economists or biologists ask: Complex organizations, systems and networks, yes, but for what purpose? One can propose the following answer: to adapt to the environment. This is the case in biology, since the Claude Bernard's "constance du milieu intérieur" and the "homeostasis" of Walter Cannon. This is naturally the case in ecology and environmental studies. This is also the case in economics when we have to adapt to scarcity constraints, balances between supply and demand, and many other ones.

The environment is described by constraints of various kinds (representing objectives, physical and economic constraints, "stability" constraints, etc.) that can never be violated. At the same time, the actions, the messages, the coalitions of actors and connectionist operators do evolve, and their evolution must be consistent with the constraints, with objectives reached at (successive) finite times (and/or must be selected through inter-temporal criteria).

There is no reason why collective constraints are satisfied at each instant by evolutions under uncertainty governed by stochastic or tychastic control dynamical systems. This leads to the study of how to correct either the dynamics, and/or the constraints in order to re-establish this consistency. This may allow us to provide an explanation of the formation and the evolution of the architecture of the system and of their variables.

Presented in such an evolutionary perspective, this approach of (complex) evolution departs from the main stream of modeling studying static networks with graph theory and dynamical complex systems by ordinary or partial differential equations, a task difficult outside the physical sciences.

For dealing with these issues, one needs concepts and formal tools, algorithms and mathematical techniques motivated by complex systems evolving under uncertainty. For instance, without entering into the details, systems sharing such common features arise in:

economics, where the viability constraints are the scarcity constraints. We can replace the fundamental Walrasian model of resource allocations by a decentralized dynamical model, in which the role of the controls is played by the prices or other economic decentralizing messages (as well as coalitions of consumers, interest rates, and so forth). The regulation law can be interpreted as the behavior of Adam Smith's invisible hand choosing the prices as a function of the allocations,

dynamical connectionist networks and/or dynamical cooperative games, where coalitions of players may play the role of controls: each coalition acts on the environment by changing it through dynamical systems. The viability constraints are given by the architecture of the network allowed to evolve,

genetics and population genetics, where the viability constraints are the ecological constraints. The state describes the phenotype and the controls are genotypes or fitness matrices.

sociological sciences, where a society can be interpreted as a set of individuals subject to viability constraints. They correspond to what is necessary to the survival of the social organization. Laws and other cultural codes are then devised to provide each individual with psychological and economical means of survival as well as guidelines for avoiding conflicts. These cultural codes play the role of regulation controls.

cognitive sciences, where at least at one level of investigation, the variables describe the sensory-motor activities of the cognitive system, while the controls translate into what could be called a conceptual control (which is the synaptic matrix in neural networks.)

control theory and differential games, conveniently revisited, can provide many metaphors and tools for grasping the above problems. Many problems in control design, stability, reachability, inter-temporal optimality, viability and capturability, observability and set-valued estimation can be formulated in terms of viability kernels. The viability kernel algorithm computes this set.

Outline: We devote the three first sections to the description of the main concepts and basic results of viability theory: evolutions, viability kernels and capture basins under evolutionary systems in the first section, characterization of viability and of the adaptation law in the second construction of static and dynamic feedbacks, including the slow and heavy ones, in the third.

We shall illustrate the main concepts and results of viability theory by revisiting the Verhulst type models in population dynamics, by providing the class of all Malthusian feedbacks (mapping states to growth rates) that guarantee the viability of the evolutions, and adapting these models to the management of renewable resources. More complex applications go beyond the format of this introduction to viability theory.

We finally address the issues related to the restoration of the viability when the constraints are not viable under an evolutionary system: correcting the dynamics by viability multipliers, re-initializing the state whenever the viability is at stake, providing hybrids of continuous and discrete time evolutions, or changing the constraints by governing their evolutions by mutational equations.

2.2. The Mathematical Framework

For more details on viability theory, we refer to Aubin (2000) in the case of differential equations, to Aubin (1991) and Aubin (1997) for the state of the art and economic applications up until 1995 and to the book Aubin et al. (2006) for recent advances.

2.2.1. Viability and Capturability

Let X denote the *state space* of the system. Evolutions describes the behavior of the state of the system as a function of time $t \in \mathbb{R}_+ := [0, \ldots, +\infty[$ ranging over the set of nonnegative real numbers or scalars $t \in \mathbb{R}_+$. We shall assume all along that

1. the state space is a finite dimensional vector space $X := \mathbb{R}^n$,
2. evolutions are *continuous* functions $x(\cdot) : t \in \mathbb{R}_+ \mapsto x(t) \in X$ describing the evolution of the state $x(t)$.

We denote the space of continuous evolutions $x(\cdot)$ by $\mathcal{C}(0,\infty;X)$ or, in short, $\mathcal{C}(X)$.

Some evolutions, mainly motivated by physics, are classical: *equilibria and periodic evolutions*. But these properties are not necessarily adequate for problems arising in economics, biology, cognitive sciences and other domains involving living beings. Hence we add the concept of evolutions *viable in a constrained set* $K \subset X$ (*the environment*) *or capturing a target* $C \subset K$ *in finite time* to the list of properties satisfied by evolutions. Therefore, we consider mainly evolutions $x(\cdot)$ *viable in a subset* $K \subset X$ representing a constrained set (an environment) in which the trajectory of the evolution must remain forever:

$$\forall\, t \geq 0,\ x(t) \in K. \tag{2.2}$$

Alternatively, a "target" $C \subset K$ being given, we distinguish evolutions $x(\cdot)$ *capturing the target* C in the sense that they are viable in K until they reach the target C *in finite time*:

$$\exists\, T \geq 0 \quad \text{such that} \quad \begin{cases} x(T) \in C \\ \forall\, t \in [0,T],\ x(t) \in K \end{cases}. \tag{2.3}$$

We devote our paper to the study of the set of evolutions viable in K outside C, i.e. that are viable in K forever or until they reach the target C in finite time.

2.2.2. The Evolutionary System

Next, we provide the mathematical description of one of the "engines" governing the evolution of the state. We assume that there exists a control parameter, or better yet a regulatory parameter, called a *regulon*, that influences the evolution the state of the system. This dynamical system takes the form of a *control system with (multi-valued) feedbacks* :

$$\begin{cases} (i) \quad x'(t) = f\big(x(t),u(t)\big) \qquad \text{(action)} \\ (ii) \quad u(t) \in U\big(x(t)\big) \qquad \text{(contingent retro-action)} \end{cases} \tag{2.4}$$

taking into account the *a priori availability* of several regulons $u(t) \in U\big(x(t)\big)$ chosen in a subset $U\big(x(t)\big) \subset Y$ of another finite dimensional vector-space Y subjected to *state-dependent constraints*.

Once the initial state is fixed, the first equation describes how the regulon *acts* on the velocities of the system whereas the second inclusion shows how the state (or an observation on the state) can *retroact through* (several) regulons in a vicariant way.

We observe that there are many evolutions starting from a given initial state x_0, one for each time-dependent regulon $t \mapsto u(t)$. The set-valued map $U : X \rightsquigarrow Y$ also describes the *state-dependent constraints on the regulons*. In this case the system (2.4) can no longer be regarded as a parameterized family of differential equations, as in the case when $U(x) \equiv U$ does not depend upon the state, but as a *differential inclusion* (see, Aubin and Cellina 1984, for example). Fortunately, differential inclusions enjoy most of the properties of differential equations. A solution to system (2.4) is an evolution $t \to x(t)$ satisfying this system for some (measurable) open-loop control $t \to u(t)$ (almost everywhere).

We associate with the control system the *evolutionary system* $x \rightsquigarrow \mathcal{S}(x)$ associating with any initial state $x \in K$ the subset $\mathcal{S}(x) \subset \mathcal{C}(0,\infty;X)$ of solutions starting at x. Most of the results on viability kernels and capture basins depend upon few properties of this evolutionary system, that are shared by other "engines of evolution", such as diffusion-reaction systems, path (or history) dependent systems, mutational equations governing the evolution of compact sets.

2.2.3. Viability Kernels and Capture Basins

The problems we will study are all related to the viability of a constrained subset K and/or the capturability of a target $C \subset K$, under the dynamical system modeling the dynamic behavior of the system.

1. The subset $\mathrm{Viab}(K)$ of initial states $x_0 \in K$ such that one solution $x(\cdot)$ to system (2.4ii) starting at x_0 is viable in K for all $t \geq 0$ is called the *viability kernel* of K under the control system. A subset K is a *repeller* if its viability kernel is empty.

2. The subset $\mathrm{Capt}(K,C)$ of initial states $x_0 \in K$ such that the target $C \subset K$ is reached in finite time before possibly leaving K by one solution $x(\cdot)$ to system (2.4ii) starting at x_0 is called the *viable-capture basin* of C in K. A subset $C \subset K$ such that $\mathrm{Capt}(K,C) = C$ is said to be *isolated* in K.

We say that

1. a subset K is *viable* under $\mathcal{S}$ if $K = \mathrm{Viab}(K)$,
2. that K is a *repeller* if $\mathrm{Viab}(K) = \emptyset$.

In other words, the *viability* of a subset K under a control system is a consistency property of the dynamics of the system, confronted to the constraints it must obey during some length of time.

To say that a singleton $\{c\}$ is viable amounts means that the state c is an *equilibrium* (equilibria, equal balance) — sometimes called a fixed point. The trajectory of a periodic solution is also viable.

Contrary to the century-old tradition going back to Lyapunov, we require the system to capture the target C in *finite time*, and not in an asymptotic way, as in mathematical models of physical systems. However, there are close mathematical links between the various concepts of *stability* and viability. For instance, Lyapunov functions can be constructed using tools of viability theory. Or one can prove that the attractor is contained in the viability kernel of an absorbing set under the backward (negative) system. This needs much more space to be described: we refer to Chapter 8 of Aubin (1991) and Chapter 8 of Aubin (1997) for more details on this topic.

One can prove that *the viability kernel* $\mathrm{Viab}(K)$ *of the subset* K *is the "largest" subset of* K *viable under the control system*. Hence, all interesting features such as equilibria, trajectories of periodic solutions, limit sets and attractors, if any, are all contained in the viability kernel.

One can prove that the viability kernel is the **unique** subset $D \subset K$ viable and isolated in K such that $K\backslash D$ is a repeller. If $K\backslash C$ is a repeller, the capture basin $\mathrm{Capt}(K,C)$ of $C \subset K$ is the **unique** subset D between C and D such that D is isolated in K and $D\backslash C$ is locally viable.

Due to viability theorems, the viability kernels of a subset and the capture basins of a target can thus be characterized in diverse ways through tangential conditions. They play a crucial role in viability theory, since many interesting concepts are often viability kernels or capture basins. See Aubin (2001b) and Aubin (2002) for more properties of viability kernels and capture basins.

Furthermore, *algorithms* designed in Saint-Pierre (1994) allow us to compute viability kernels and capture basins (see, e.g., Cardaliaguet et al. 1999, Quincampoix and Saint-Pierre 1998). In general, there are no explicit formulas providing the viability kernel and capture basins.

2.3. Characterization of Viability and/or Capturability

The main task is to characterize the subsets having this viability/capturability property. To be of value, this task must be done without solving the system for checking the existence of viable solutions for each initial state.

2.3.1. Tangent Directions

An immediate intuitive idea jumps to mind: at each point on the boundary of the constrained set outside the target, where the viability of the system is at stake, there should exist a velocity which is in some sense *tangent* to the viability domain- it serves to allow the solution to bounce back and remain inside it. This is, in essence, what the viability theorem states. However, before stating it, the mathematical implementation of the concept of tangency must be made.

We cannot be content with viability sets that are smooth manifolds (such as spheres, which have no interior), because inequality constraints would thereby be ruled out (as for balls, that possess distinct boundaries). Therefore we need to implement the concept of a direction v tangent to K at $x \in K$, which should mean that starting from x in the direction v, we do not go too far from K: The adequate definition due to G. Bouligand and F. Severi proposed in 1930 states that a direction v is *tangent to* K *at* $x \in K$ if it is a limit of a sequence of directions v_n such that $x + h_n v_n$ belongs to K for some sequence $h_n \to 0+$. The collection of such directions, which are in some sense inward, constitutes a closed cone $T_K(x)$, called the *tangent cone*[2] to K at x. Naturally, except if K is a smooth manifold, we lose the fact that the set of tangent vectors is a vector-space. However, this discomfort is not unbearable,

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[2] Replacing the linear structure underlying the use of tangent spaces by the tangent cone is at the root of *set-valued analysis*.

since advances in set-valued analysis built a calculus of these cones allowing us to compute them (see, e.g., Aubin and Frankowska 1990, Rockafellar and Wets 1997).

2.3.2. The Adaptive Map

We then associate with the dynamical system (described by (f,U)) and with the viability constraints (described by K) the *(set-valued) adaptive or regulation map* R_K. It maps any state $x \in K\backslash C$ to the subset $R_K(x)$ (possibly empty) consisting of regulons $u \in U(x)$ which are *viable* in the sense that $f(x,u)$ is tangent to K at x:

$$R_K(x) := \{u \in U(x) \mid f(x,u) \in T_K(x)\}.$$

We can, for example, *compute the adaptive map* in many instances.

2.3.3. The Viability Theorem

The viability theorem states that *the target C can be reached in finite time from each initial condition $x \in K\backslash C$ by at least one evolution of the control system viable in K, if and only if, for every $x \in K\backslash C$, there exists at least one viable control $u \in R_K(x)$.*

This viability theorem holds true when both C and K are closed and for a rather large class of systems, called *Marchaud systems*: beyond imposing some weak technical conditions, the only severe restriction is that, for each state x, the set of velocities $f(x,u)$ when u ranges over $U(x)$ is *convex* (this happens for the class of control systems of the form

$$x'(t) \; = \; f\big(x(t)\big) + G\big(x(t)\big)\, u(t)\,,$$

where $G(x)$ are linear operators from the control space to the state space, when the maps $f : X \mapsto X$ and $G : X \mapsto \mathcal{L}(Y,X)$ are continuous and when the control set U (or the images $U(x)$) are convex.

Curiously enough, viability implies stationarity, i.e., the existence of an equilibrium. Equilibria being specific evolutions, their existence requires stronger assumptions. The equilibrium theorem states that *when the constrained set is assumed to be viable, convex and compact, there exists a (viable) equilibrium*. Without convexity, we deduce only the existence of minimal viable closed subsets.

The proofs of the above viability theorem and the equilibrium theorem are difficult: The equilibrium theorem is derived from the 1910 Brouwer fixed point theorem, and the proof of the viability theorem uses all the theorems of functional analysis except the closed graph theorem and the Lebesgue convergence theorem. However, their consequences are much easier to obtain and can be handled with moderate mathematical competence.

2.3.4. The Adaptation Law

Once this is done, and whenever a constrained subset is viable for a control system, the second task is to show how to govern the evolution of viable evolutions. We thus prove that viable evolution of system (2.4) are governed by

$$\begin{cases} (i) & x'(t) = f(x(t),u\big(t\big)) \\ (ii) & u(t) \in R_K\big(x(t)\big) \qquad \text{(adaptation law)} \end{cases} \tag{2.5}$$

until the state reaches the target C. We observe that the initial set-valued map U involved in (2.4ii) is replaced by the adaptive map R_K in (2.5ii). The inclusion $u(t) \in R_K(x(t))$ can be regarded as an *adaptation law* (rather than a learning law, since there is no storage of information at this stage of modeling).

2.3.5. Planning Tasks: Qualitative Dynamics

Reaching a target is not enough for studying the behavior of control systems that have to *plan* tasks in a given order. This issue has been recently revisited in Aubin and Dordan (2002), in the framework of qualitative physics (see, e.g., Kuipers 1994; Dordan 1995; Aubin 1996; Eisenack and Petschel-Held 2002; Eisenack et al. 2006, and Chapter 3 in this book, for more details on this topic). We describe the sequence of tasks or objectives by a family of subsets regarded as qualitative cells. Giving an order of visit of these cells, the problem is to find an evolution visiting these cells in the prescribed order.

2.3.6. The Meta-System

In order to bound the chattering (rapid oscillations or discontinuities) of the regulons, we set *a priori* constraints on the velocities of the form

$$\forall\, t \geq 0,\ \ \|u'(t)\| \ \leq\ c\,.$$

Let $B(0,c) \subset Y$ denote the ball of radius c centered at the origin. The bound on the velocity of the regulons is taken into account by the *meta-system*. Associating it with the initial viability problem is the system:

$$\left\{ \begin{array}{lll} (i) & x'(t) \ = \ f\big(x(t),u(t)\big) \\ (ii) & u'(t) \ \in \ B(0,c) \end{array} \right. \tag{2.6}$$

subjected to the *meta-constraints*

$$\forall\, t \geq 0,\ \ x(t) \in K \text{ and } u(t) \in U\big(x(t)\big)\,. \tag{2.7}$$

Unfortunately, the above meta-constraints may no longer be viable under the meta-system.

2.4. Selecting Viable Feedbacks

2.4.1. Static Feedbacks

A *(static) feedback* r is a map $x \in K \mapsto r(x) \in X$ which is used to pilot evolutions governed by the differential equation $x'(t) = f\big(x(t), r(x(t))\big)$. A *feedback* r is said to be *viable* if the solutions to the differential equation $x' = f\big(x, r(x)\big)$ are viable in K. The most celebrated examples of linear feedbacks in linear control theory, designed to control a system, are not viable for an arbitrary constrained set K and, according to the constrained set K, the viable feedbacks are not necessarily linear.

However, the viability theorem implies that a feedback r is viable if and only if r is a *selection* of the adaptive map R_K in the sense that

$$\forall\, x \in K \backslash C,\ \ r(x)\ \in\ R_K(x)\,. \tag{2.8}$$

Hence, the method for designing feedbacks for control systems in order to evolve in a constrained subset, amounts to find *selections* $r(x)$. One can design a factory for designing selections (see, Chapter 6 of Aubin (1991), for instance). Ideally, a feedback should be continuous to guarantee the existence of a solution to the differential equation $x' = f\big(x, r(x)\big)$. But this is not always possible. This is the case of *slow selection* r° of R_K of minimal norm, governing the evolution of *slow viable evolutions* (despite its lack of continuity).

2.4.2. Dynamic Feedbacks

One can also look for *dynamic feedbacks* $g_K : K \times Y \mapsto Y$ that governs the evolution of both the states and the regulons through the *meta-system* of differential equations

$$\left\{ \begin{array}{lll} (i) & x'(t) \ = \ f\big(x(t), u(t)\big) \\ (ii) & u'(t) \ = \ g_K\big(x(t), u(t)\big)\,. \end{array} \right. \tag{2.9}$$

A *dynamic feedback* g_k *is viable* if the meta-constraints (2.7) are viable under the meta-system (2.9).

As for the (static) feedbacks, one can prove that all the viable dynamic feedbacks are selections of a *dynamical adaptive map* $G_K : K \times Y \rightsquigarrow Y$ obtained by differentiating the adaptation law (2.4ii), due to the differential calculus of set-valued maps (see, Aubin and Frankowska 1990).

2.4.3. Heavy Evolutions and the Inertia Principle

Among the viable dynamic feedbacks, one can choose the *heavy viable dynamic feedback* $g_K^{\circ} \in G_K$ with minimal norm that governs the evolution of *heavy viable solutions*, i.e., viable evolutions with *minimal velocity*. They are called *heavy viable evolutions*[3] in the sense of heavy trends in economics.

Heavy viable evolutions offer convincing metaphors of the evolution of biological, economic, social and control systems that obey the *inertia principle*. It states in essence that *the regulons are kept constant as long as viability of the system is not at stake.* Heavy viable evolutions can be viewed as providing mathematical metaphors for the concept of *punctuated equilibrium* introduced in paleontology by Eldredge and Gould (1972). In our opinion, this is a mode of regulation of control systems (see, Chapter 8 of Aubin (1996) for further justifications).

Indeed, as long as the state of the system lies in the interior of the constrained set (i.e., away of its boundary), any regulon will do. Therefore, the system can maintain the regulon inherited from the past. This happens if the system obeys the inertia principle. Since the state of the system evolves while the regulon remains constant, it may reach the viability

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[3] When the regulons are the velocities, heavy solutions are the ones with minimal acceleration, i.e., maximal inertia.

boundary with an outward velocity. This event corresponds to a period of *viability crisis*:in order to survive, the system must find other regulons such that the new associated velocity forces the solution back, inside the viability set until the time when a regulon can remain constant for some time.

2.5. Management of Renewable Resources

Let us consider the regulons of the system (2.1)

$$x'(t) \;=\; u(t)\,x(t)$$

chosen to govern evolutions viable in the interval $[a,b]$. The *equilibrium map* $U_\propto$ is defined by $U_\propto(x) = \{0\}$ and the *monotonic maps* U_+ and U_- by

$$U_+(x) \;:=\; \mathbb{R}_+ \;\text{ and }\; U_-(x) \;:=\; \mathbb{R}_- \,.$$

The regulation map is equal to

$$R_K(x) \;:=\left\{\begin{array}{lll} \mathbb{R}_+ & \text{if} & x \;=\; a \\ \mathbb{R} & \text{if} & x \;\in]a,b[\\ \mathbb{R}_- & \text{if} & x \;=\; b\,. \end{array}\right.$$

It is set-valued, has non-empty values, but has too poor continuity property, a source of mathematical difficulties. *The interval* $[a,b]$ *is obviously viable under such a control system.*

As mentioned in Section 2.1.1., we regard the growth rates as regulons of the system (2.1): $x'(t) = u(t)x(t)$.

The affine feedback map $r(b-x)$ providing the Pearl-Verhulst logistic equation is always positive on the interval $[a,b]$, so that the velocity of the population is always non-negative, even though the population slows down. In order to have negative velocities (still with positive growth rates), we should require that the feedback satisfies the following phenomenological properties: for some $\xi \in [a,b[$,

$$\left\{\begin{array}{ll} (i) & \forall\, x \in [a,\xi[,\; \widetilde{u}'(x) \;>\; 0 \\ (ii) & \forall\, x \in]\xi,b],\; \widetilde{u}'(x) \;<\; 0 \\ (iii) & u(b) \;=\; 0 \end{array}\right. .$$

The increasing behavior of $\widetilde{u}(x)$ on the interval $[a,\xi[$ is called the *allee effect*, stating that at a low population size, an increase of the population size is desirable and has positive effects on population growth, whereas the decreasing behavior of $\widetilde{u}(x)$ on the interval $]\xi,b]$ is called the *logistic effect*, stating that at high population, an increase of the size has a negative effect on the growth of the population.

Inert evolutions are obtained by taking the maximal velocities allowed: $u'(t) = \pm c$. Heavy evolutions combine Malthusian and inert growth. They are obtained starting with constant growth rate (Malthusian evolution) of the state (by taking $u'(t) = 0$ until the time when the inertia bound is met (and thus, implying a (weak) allee effect during this phase of the evolution). This provides *warning signals telling when, where and how the regulons must evolve*: The feedback becomes a specific one, the inert feedback, providing constant (negative) velocities $u'(t) = -c$ of growth rates driving the state at its carrying capacity b.

2.5.1. Discrete Versus Continuous Time

Discrete evolutions are maps associating with each discrete time $j \in \mathbb{N} := \{0,\ldots,+\infty\{$ ranging over the set of non-negative integers a state $x(j) =: x_j \in X$. Unfortunately, for discrete time evolutions, tradition imposes upon us to regard discrete evolutions as sequences and to use the notation $\overrightarrow{x} : j \in \mathbb{N} \mapsto x_j := x(j) \in X$.

The choice between these two representations of time, the discrete and the continuous, is not easy. The natural one, that appears the simplest for the non-mathematicians, is the choice of the set $\mathbb{N}$ of discrete times. However, there are drawbacks. On the one hand, it may be difficult to find a common *time scale* for the different components of the state variables of the state space of a given type of models. On the other hand, by doing so, we deprive ourselves of the concepts of velocity, acceleration and other dynamical concepts that are not well taken into account by discrete time systems as well as of the many results of the differential and integral calculus gathered for more than four centuries. However, for computational purposes, we shall approximate continuous-time systems by discrete time ones where the time scale becomes infinitesimal. However, *viability properties of the discrete analogues of continuous-time systems can be drastically different*: we shall see in the simple example of the Verhulst logistic equation that the interval $[0,1]$ is invariant under the continuous system

$$x'(t) \;=\; rx(t)\big(1-x(t)\big)\,,$$

whereas the viability kernel of $[0,1]$ under its discrete analog

$$x_{n+1} \;=\; rx_{n+1}(1-x_{n+1})$$

is a Cantor subset of $[0,1]$ when $r>4$. See Fig. 2.1.

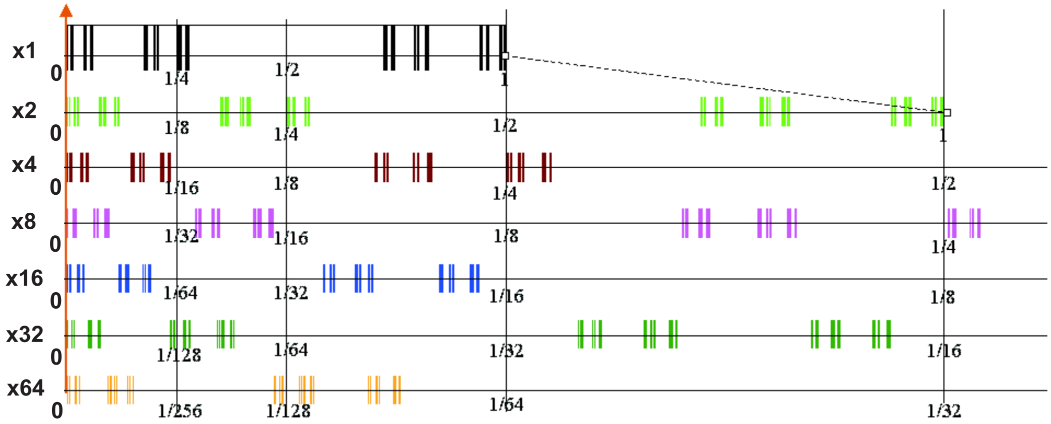

Figure 2.1. Viability kernel of $[0,1]$ under the quadratic map: The interval $[0,1]$ has been magnified several times for easier visibility and for highlighting the Cantor structure of the viability kernel.

2.5.2. Introducing Inertia Bounds

The constraint on the velocities of the growth rates imposed by the inertia property means that

$$|u'(t)| \leq c .$$

This suggests to take the velocity of the regulons as "meta-regulons", required to range over the interval $[-c,+c]$. The state-regulon pairs (x,u) are the "meta-states" of the "meta-system":

$$\begin{cases} (i) & x'(t) = u(t)\,x(t) \\ (ii) & |u'(t)| \leq c \end{cases} \tag{2.10}$$

where the meta-regulon is chosen to be the velocity or the growth rate of the system. Unfortunately, the meta-constrained set $[a,b]\times\mathbb{R}_+$ is obviously not viable under the above meta-system: Every solution starting from (a,u) with $u<0$ leaves the set $[a,b]\times\mathbb{R}_+$ immediately, as do evolutions starting from (b,u) with $u>0$. We thus define the graph of the set-valued map U_c by

$$\text{Graph}(U_c) := \text{Viab}_{(2.10)}\big([a,b]\times\mathbb{R}_+\big).$$

The regulation map U_c for the one-dimensional model in Eq. (2.1) can be explicitly described by two feedbacks maps: we associate the inert feedbacks

$$r^{\sharp}(x) := \sqrt{2\log\left(\frac{b}{x}\right)},\ \ r^{\flat}(x) := \sqrt{2\log\left(\frac{x}{a}\right)}, \quad \text{and} \quad R(x) := \left[-r^{\flat}(x),+r^{\sharp}(x)\right].$$

We can prove that for system $x'(t)=u(t)x(t)$, the graph of the regulation map U_c is limited by the graphs of $-\sqrt{c}\,r^{\flat}$ below and $\sqrt{c}\,r^{\sharp}$ above: The regulation map is equal to

$$U_c(x) := \sqrt{c}\,\left[-r^{\flat}(x),+r^{\sharp}(x)\right] = \sqrt{c}\,R(x).$$

The graph of U_c can also be computed by the Saint-Pierre viability kernel algorithm, as it is shown in Fig. 2.2.

We shall now construct several feedbacks as selections of the regulation map U_c.

2.5.2.1. Affine Feedbacks and the Verhulst Logistic Equation

Affine feedbacks defined by

$$\widetilde{u}(x) := r(b-x),$$

are selections of the regulation map U_c when $r \leq \sqrt{c}\,\frac{2}{b}$, so that *the viability of the interval* $[a,b]$ *under the models using such affine feedbacks is guaranteed*. The Verhulst logistic differential equation $x'(t)=rx(t)(b-x(t))$ corresponds to such an affine feedback. The solutions starting from $x\in[a,b]$ are equal to

$$x(t) = \frac{bx}{x+(b-x)e^{-rt}}.$$

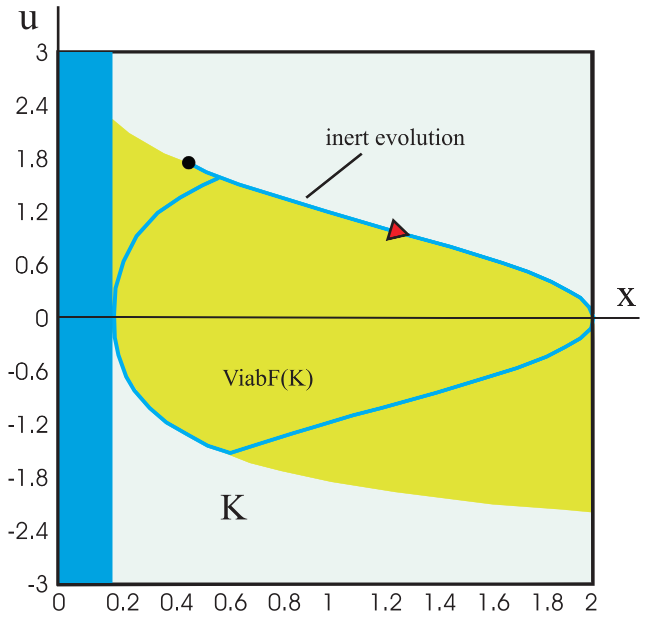

Figure 2.2. Viability kernel and inert evolution. Courtesy of Patrick Saint-Pierre.

They remain confined in the interval $[a,b]$ and converge to the equilibrium b when $t \mapsto +\infty$. We observe that the velocity and the growth rate $u(t)$ are respectively given by

$$u(t) \;=\; re^{-rt}(b-x) \qquad \text{and} \qquad u(t) \;=\; \frac{r(b-x)e^{-rt}}{x+(b-x)e^{-rt}}.$$

They start $\frac{r(b-x)}{b}$, converge to 0 when $t \mapsto +\infty$ and are always decreasing since their derivatives

$$u'(t) \;=\; \frac{-r^2(b-x)e^{-rt}}{(x+(b-x)e^{-rt})^2}$$

are negative.

Remark: Other Example of Feedback — an explicit feedback used in population dynamics, defined by

$$\widetilde{u}(x) \;:=\; r\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{b}}\right)$$

which also produces only a logistic effect, the derivative $\widetilde{u}'(x)$ being always non-positive.

2.5.2.2. Inert Evolutions

For a given inertia bound c, inert evolutions are governed by the feedbacks $\overline{u}(t) = \sqrt{c}\, r^{\sharp}(x(t))$ if $u > 0$ and $\overline{u}(t) = -\sqrt{c}\, r^{\flat}(x(t))$ if $u < 0$. See Fig. 2.2.

Let us consider the case when $u > 0$ (the case when $u < 0$ being symmetrical). Then the derivative of the feedback $\sqrt{c}\, r^{\sharp}$ being negative on $[a,b]$, it is a purely logistic feedback with no allee effect. In both cases, the velocity governing the inert evolution is constant and equal to $\overline{u}'(t) = -c$, so that $\overline{u}(t) = u - ct$. Therefore the evolution of the inert regulon is given by

$$\overline{u}(t) \;=\; u\left(1 - \frac{ut}{2\log\left(\frac{b}{x}\right)}\right)$$

and the evolution of the inert state by

$$\overline{x}(t) \;=\; xe^{ut - \frac{u^2t^2}{4\log\left(\frac{b}{x}\right)}}.$$

The state reaches the equilibrium $(b,0)$ at time

$$\tau(x,u) \;=\; 2\frac{\log\left(\frac{b}{x}\right)}{u}.$$

Then, taking $\overline{x}(t) \equiv b$ and $\overline{u}(t) \equiv 0$, the solution remains at equilibrium for $t \geq \tau(x,u)$.

The derivative of the feedback $\overline{u}(x) := \sqrt{c}\, r^{\sharp}(x)$ governing the evolution of the inert evolution being negative whenever $a \leq x < b$, *the inert evolution does not show any allee effect, but only a logistic one.*

This is the same situation as with the Verhulst equation. For obtaining feedbacks conveying the allee effect, we introduce heavy evolutions associated with a bound $c > 0$ on the velocities of the regulons.

2.5.2.3. Heavy Evolutions

Heavy solutions x_c are obtained when the regulon is kept constant as long as possible. Starting from (x,u), the state $x_c(\cdot)$ of the heavy solutions evolves according to

$$x_c(t) \;=\; xe^{ut}$$

and reaches b at time $\frac{\log\left(\frac{b}{x}\right)}{u}$ (half the time needed for the inert evolution to reach equilibrium) with velocity equal to $u > 0$ and $ub > 0$, so that $x_c(\cdot)$ leaves the interval $[a,b]$ in finite time. This evolution reaches the boundary of the graph of U_c at

$$\begin{cases} \textbf{warning state} & \xi_c(x,u) \;=\; be^{-\frac{u^2}{2c}} \\ \textbf{warning time} & \sigma_c(x,u) \;:=\; \frac{\log\left(\frac{b}{x}\right)}{u} - \frac{u}{2c}. \end{cases}$$

Hence, once a velocity limit c is fixed, the heavy solution evolves with constant regulon u until the last instant $\sigma_c(x,u)$ when the state reaches $\xi_c(x,u)$ and the velocity of the regulon $\alpha\big(\xi_c(x,u),u\big) = c$. *This is the last time when the regulon* has to change by taking

$$u_c(t) \;=\; u - c\left(t - \frac{\log\left(\frac{b}{x}\right)}{u} + \frac{u}{2c}\right)$$

so that the evolution follows the inert solution starting at $\big(\xi_c(x,u),u\big)$. It reaches equilibrium $(b,0)$ at time $t^\star := \sigma_c(x,u) + \frac{u}{2c}$:

$$t^\star \;:=\; \frac{\log\left(\frac{b}{x}\right)}{u} + \frac{u}{2c}$$

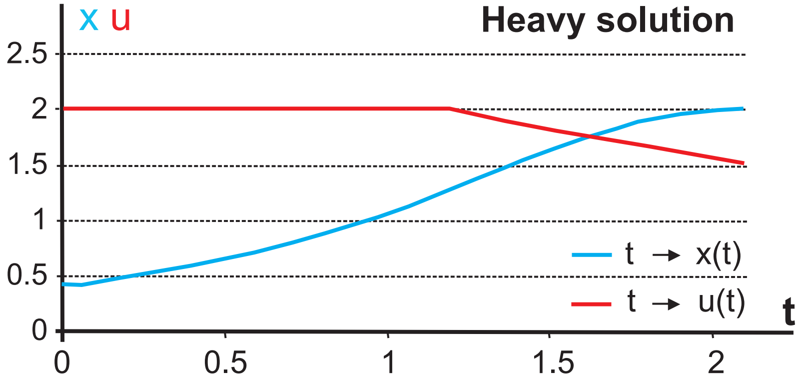

Figure 2.3. Graph of the heavy evolution. Both the graphs of the heavy evolution (in blue) and of its control (in red) are plotted. They are not computed from the analytical formulas given below, but extracted from the viability kernel algorithm. The control remains constant until the trajectory of the exponential solution hits the boundary of the viability kernel and then slows down when it is controlled with a decreasing linear time dependent controls with velocity equal to $-c$. It reaches in finite time the boundary of the constrained interval with a velocity equal to 0 and may remain at this equilibrium. Courtesy of Patrick Saint-Pierre.

with an advance equal to $\sigma_c(x,u)$ over the inert solution. The *heavy evolution* $(x_c(\cdot),u_c(\cdot))$ is associated with the heavy feedback $\widetilde{u_c}$ defined by

$$\widetilde{u_c}(y) := \begin{cases} u & \text{if} \quad x \le y \le \xi_c(x,u) \\ \sqrt{c}\, r^{\sharp}(y) & \text{if} \quad \xi_c(x,u) \le y < b \\ 0 & \text{if} \quad y = b \end{cases} .$$

See Fig. 2.2. *The heavy feedback has an allee effect on the interval* $[x,\xi_c(x,u)]$ *and a logistic effect on* $[\xi_c(x,u),b]$. *Bounding the inertia by c, the feedback governing the heavy evolution maximizes the allee effect.*

Indeed, we observe that the part of the graph of any feedback $\widetilde{u}$ passing through (x,u) $(u = \widetilde{u}(x))$ has an allee effect only when it lies above the horizontal line passing through (x,u). It necessarily intersects the graph of $\sqrt{c}\, r^{\sharp}$ before reaching $(\xi_c(x,u),u)$. The allee effect of the heavy evolution is weak in the sense that the velocity of the regulon is equal to 0 instead of being strictly positive. But it lasts longer (cf. Fig. 2.3).

In summary, the heavy evolution under bound $c > \alpha(x,u)$ is described by the following formulas: the regulons are equal to

$$u_c(t) = \begin{cases} u & \text{if} \quad t \in \left[0, \frac{\log\left(\frac{b}{x}\right)}{u} - \frac{u}{2c}\right] \\ u - c\left(t - \frac{\log\left(\frac{b}{x}\right)}{u} + \frac{u}{2c}\right) & \\ & \text{if} \quad t \in \left[\frac{\log\left(\frac{b}{x}\right)}{u} - \frac{u}{2c}, \frac{\log\left(\frac{b}{x}\right)}{u} + \frac{u}{2c}\right] \end{cases} ,$$

and the states to $x_c(t) \; =$

$$\begin{cases} xe^{ut} & \text{if} \quad t \in \left[0, \frac{\log\left(\frac{b}{x}\right)}{u} - \frac{u}{2c}\right] \\ be^{-\frac{u^2}{2c}+u\left(t-\frac{\log\left(\frac{b}{x}\right)}{u}+\frac{u}{2c}\right)-\frac{u^2\left(t-\frac{\log\left(\frac{b}{x}\right)}{u}+\frac{u}{2c}\right)^2}{2}} & \\ & \text{if} \quad t \in \left[\frac{\log\left(\frac{b}{x}\right)}{u} - \frac{u}{2c}, \frac{\log\left(\frac{b}{x}\right)}{u} + \frac{u}{2c}\right] . \end{cases}$$

Remark: — Let $f : [a,b] \mapsto [0,\infty[$ be a function such that the primitives $F_a(x) := \int_a^x \frac{dx}{f(x)}$ and $F_b(x) := \int_x^b \frac{dx}{f(x)}$ exist. In the above examples, we took $f(x) = x$, so that $F_a(x) = \log\left(\frac{x}{a}\right)$ and $F_b(x) = \log\left(\frac{b}{x}\right)$. We can adapt the previous formulas to the case of control systems of the form

$$x'(t) \; = \; u(t)f\big(x(t)\big)$$

defined on the interval $[a,b]$. The corresponding feedbacks $r^\sharp$ and $r^\flat$ are given by

$$r^\sharp(x) \; := \sqrt{2F_b(x)} \qquad \text{and} \qquad r^\flat(x) \; := \sqrt{2F_a(x)} \qquad ,$$

the derivatives of which are given by

$$r^{\sharp\,\prime}(x) \; := \; \frac{-1}{r^\sharp(x)f(x)} \qquad \text{and} \qquad r^{\flat\,\prime}(x) \; := \; \frac{1}{r^\flat(x)f(x)} \, .$$

The regulation map U_c is equal to

$$U_c(x) \; := \sqrt{c}\; \left[-r^\flat(x), +r^\sharp(x)\right] ,$$

so that the meta-viability constrained set $[a,b] \in \mathbb{R}$ is viable under the control system

$$x'(t) \; = \; u(t)f\big(x(t)\big) \text{ where } u'(t) \; \in \; U_c\big(x(t)\big)$$

under bounded inflation. We may choose among the evolutions governed by this system the inert evolution, or the heavy evolution, maximizing the allee effect on the interval for a given inertia bound c as we did for $x'(t) = u(t)x(t)$.

2.5.2.4. The Heavy Hysteresis Cycle

The heavy evolution $(x_c(\cdot), u_c(\cdot))$ starting at (x,u) where $u > 0$ stops when reaching the equilibrium (b,u) at time $t^\star := \sigma_c(x,u) + \frac{u}{2c}$.

But as (b,u) lies on the boundary of $[a,b] \times \mathbb{R}$, there are (many) other possibilities to find evolutions starting at $(b,0)$ remaining viable while respecting the velocity limit on the regulons. For instance, for $t \geq t^\star$, we keep the evolutions $x_h(t)$ defined by

$$\forall\, t \geq t^\star, \;\; x_h(t) \; = \; xe^{ut-\frac{u^2t^2}{4\log\left(\frac{b}{x}\right)}}$$

associated with the regulons

$$\forall\, t \geq t^\star,\ \ u_h(t) \ = \ u\left(1 - \frac{ut}{2\log\left(\frac{b}{x}\right)}\right).$$

The meta-state $(x_h(\cdot),u_h(\cdot))$ ranges over the graph of the map $-\sqrt{c}\,r^\sharp$. Therefore the regulon $u_h(t)$ and the state velocity $x_h'(t)$ become negative, so that the population size $x_h(t)$ starts decreasing. The velocity of the negative regulon is constant. But it is no longer viable on the interval $[a,b]$, because such a strictly negative velocity, $x_h(\cdot)$ leaves $[a,b]$ in finite time. Hence regulons have to be switched before the evolution leaves the graph of U_c by crossing through the graph of $-\sqrt{c}\,r^\flat$.

Thus letting the heavy solution bypass the equilibrium by keeping its velocity equal to $-c$ instead of switching it to 0, allows us to build a periodic evolution by taking velocities of regulons equal successively to 0, $-c$, 0, $+c$, and so on. We obtain in this way a periodic evolution showing an *hysteresis property*: *The evolution oscillates between a and b back and forth by ranging alternatively two different trajectories on the meta-state space* $X \times \mathcal{U}$. More precisely, we introduce the following notations: Denote by $a_c(u)$ and $b_c(u)$ the roots

$$a_c(u) \ = \ ae^{\frac{u^2}{2c}} \qquad \text{and} \qquad b_c(u) \ = \ be^{-\frac{u^2}{2c}}$$

of the equations $r^\flat(x) = u$ and $r^\sharp(x) = u$. Then $x^\star := a_c(u) = b_c(u)$, if and only if $u :=\sqrt{c}\,u^\star$ where

$$x^\star :=\sqrt{ab} \text{ and } u^\star :=\sqrt{\log\left(\frac{b}{a}\right)}.$$

Therefore $a_c(u) \leq b_c(u)$ if and only if $u \leq\sqrt{c}u^\star$. We also set

$$\tau^\star(u) \ = \ 2\frac{\log\left(\frac{b}{a}\right)}{u}.$$

The *periodic heavy hysteresis cycle* $x_h(\cdot)$ (of period $2\tau^\star(u) + \frac{3u}{c}$) is described in the following way:

1. The meta-state $\big(x_h(\cdot),u_h(\cdot)\big)$ starts from $\big(a_c(u),u\big)$ by taking the velocity of the regulon equal to 0. It remains viable on the time interval $[0,\tau^\star(u) - \frac{u}{2c}]$ until it reaches the meta-state $(b_c(u),u)$.

2. The meta-state $\big(x_h(\cdot),u_h(\cdot)\big)$ starts from $\big(b_c(u),u\big)$ at time $\tau^\star(u) - \frac{u}{2c}$ by taking the velocity of the regulon equal to $-c$. It ranges over the graph of $\sqrt{c}\,r^\sharp$ on the time interval $[\tau^\star(u) - \frac{u}{2c}, \tau^\star(u) + \frac{3u}{2c}]$ until it reaches the meta-state $\big(b_c(u),-u\big)$.

3. The meta-state $\big(x_h(\cdot),u_h(\cdot)\big)$ starts from $\big(b_c(u),-u\big)$ at time $\tau^\star(u) + \frac{3u}{2c}$ by taking the velocity of the regulon equal to 0. It remains viable on the time interval $[\tau^\star(u) + \frac{3u}{2c}, 2\tau^\star(u) + \frac{u}{c}]$ until it reaches the meta-state $(a_c(u),-u)$.

4. The meta-state $\big(x_h(\cdot),u_h(\cdot)\big)$ starts from $(a_c(u),-u)$ at time $2\tau^\star(u) + \frac{u}{c}$ by taking the velocity of the regulon equal to $+c$. It ranges over the graph of $\sqrt{c}\,r^\flat$ on the time interval $[2\tau^\star(u) + \frac{u}{c}, 2\tau^\star(u) + \frac{3u}{c}]$ until it reaches the meta-state $(a_c(u),u)$.

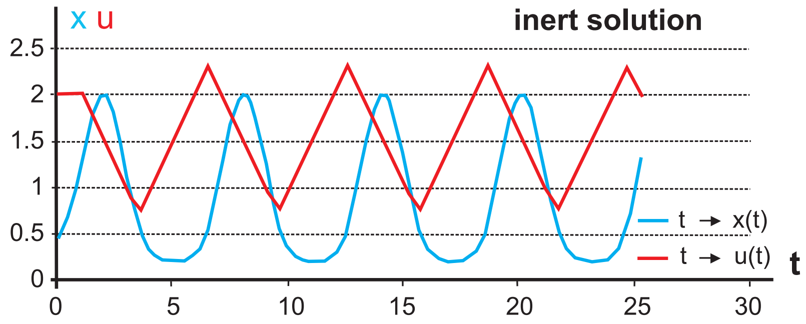

Figure 2.4. Graph of the inert evolution. Both the graphs of the inert evolution (in blue) and of its control (in red) are plotted. They are not computed from the analytical formulas given below, but extracted from the viability kernel algorithm. The velocity of the control remains constant until the trajectory of the solution hits the boundary of the viability kernel, then switches to the other extremal control with opposite sign, and so on. The evolution is then periodic, alternatively increasing and decreasing from the lower bound of the constrained interval to its upper bound. Source: Patrick Saint-Pierre.

For the limiting case when $u := \sqrt{c}\,u^\star$, it becomes the *inert hysteresis cycle* $x_h(\cdot)$ (of period $\frac{4u^\star}{\sqrt{c}}$) described in the following way (cf. Fig. 2.4):

1. The meta-state $\big(x_h(\cdot), u_h(\cdot)\big)$ starts from $(x^\star, \sqrt{c}\,u^\star)$ at time 0 with the velocity of the regulon equal to $-c$. It ranges over the graph of $\sqrt{c}\,r^\sharp$ on the time interval $[0, \frac{2u^\star}{\sqrt{c}}]$ until it reaches the meta-state $(x^\star, -\sqrt{c}\,u^\star)$,
2. The meta-state $\big(x_h(\cdot), u_h(\cdot)\big)$ starts from $(x^\star, -\sqrt{c}\,u^\star)$ at time $\frac{2u^\star}{\sqrt{c}}$ by taking the velocity of the regulon equal to $+c$. It ranges over the graph of $\sqrt{c}\,r^\flat$ on the time interval $[\frac{2u^\star}{\sqrt{c}}, \frac{4u^\star}{\sqrt{c}}]$ until it reaches the meta-state $(x^\star, \sqrt{c}\,u^\star)$.

2.5.3. Verhulst and Graham

Biologists and ecologists are rather interested in long horizons and survival or viability problems, whereas economists are mostly preoccupied with short horizons, concentrating on efficiency and substitutability of commodities (see, Gabay 1994 for this basic point).

The simplest ecological models are based on the logistic dynamics proposed by Verhulst. This evolution is slowed down by economic activity which depletes it. The basic model was originated by Graham (1935) and taken up by Schaeffer (1954), where it is assumed that the exploitation rate is proportional to the biomass and the economic activity. The equilibria of such dynamics are called *sustainable yields*. Often the purpose of investigations is to find an equilibrium yield maximizing some profit function. Beyond this static approach, optimal inter-temporal optimization was used in economics for proposing "bang-bang" solutions, which may not be viable economically, but which bypass the inertia principle, translating many rigidities in the way of operating the capture of the renewable resource.

We present here a qualitative study by Luc Doyen and Daniel Gabay (see, Doyen and Gabay 1996 for more details) dealing with the viability of both economic and ecological constraints (see, also Scheffran 2000; Eisenack et al. 2006 and Chapter 3 in this book for further discussions on these issues).

We denote by $x \in \mathbf{R}_+$ the biomass of the renewable resource and by $v \in \mathbf{R}_+$ the economic effort for exploiting it, playing the role of the regulon. The state and the regulon is subjected to viability constraints of the form

1. *Ecological constraints*
$$\forall\, t \geq 0,\ 0 \ \leq\ x(t)\ \leq\ b\,,$$
where b is the carrying capacity of the resource.
2. *Economic constraints*
$$\forall\, t \geq 0,\ cv(t) + C\ \leq\ \gamma v(t)\, x(t)\,,$$
where $C \geq 0$ is a fixed cost, $c \geq 0$ the unit cost of economic activity and $\gamma \geq 0$ the price of the resource.
3. *Production constraints*
$$\forall\, t \geq 0,\ 0\ \leq\ v(t)\ \leq\ \overline{v},$$
where $\overline{v}$ is maximal exploitation effort.

We assume that
$$\gamma b\ >\ c \quad \text{and} \quad \frac{C}{\gamma b - c}\ \leq\ \overline{v}.$$

Therefore, setting $a := \dfrac{C + c\overline{v}}{\gamma\overline{v}}$, the economic constraint implies that
$$\forall\, t \geq 0,\ x(t)\ \in\ [a,b]\,.$$

The above constraints are summarized under the set-valued map $V : [a,b] \rightsquigarrow \mathbb{R}_+$ defined by
$$\forall\, x \in [a,b],\ V(x)\ :=\ \left[\frac{C}{\gamma x - c}, \overline{v}\right].$$

The dynamics involve the Verhulst logistic dynamics and the Schaeffer proposal:
$$\begin{cases} (i) \quad x'(t)\ =\ rx(t)\left(1 - \dfrac{x(t)}{b}\right) - v(t)x(t) \\ (ii) \quad v(t)\ \in\ V(x(t))\ :=\ \left[\dfrac{C}{\gamma x(t) - c}, \overline{v}\right] \end{cases} \qquad . \tag{2.11}$$

The "equilibrium curve" here is the union of the singleton $\{0,0\}$ and the graph of the Verhulst feedback $\widetilde{v}(x)\ =\ r\left(1 - \dfrac{x}{b}\right)$. They are called "sustainable yields" in the literature. They are stable and viable if the graph of $\widetilde{v}(x)\ :=\ \dfrac{C}{\gamma x - c}$ intersects the equilibrium line: viable equilibrium belong to the interval $[x_-, x_+]$ where $x_\pm$ are the real roots of the equation
$$\gamma r x^2 - rx(\gamma b - c) + (C + rc)b\ =\ 0\,.$$

Setting $v_{\pm} := r\left(1-\frac{x_{\pm}}{b}\right)$, we can see that $[a,b]$ is viable whenever $v_{-} \geq \overline{v}$, i.e., if and only if the growth rate r is large enough:

$$r \;\geq\; \frac{b\gamma\overline{v}^2+rC}{b\gamma-c} \quad .$$

Otherwise, its easy to check that the viability kernel of the interval $[a,b]$ under system (2.11) is equal to the interval $[x_{-},x_{+}]$. We set a bound to the velocity of the economic effort, which translates the rigidity of the economic behavior:

$$\forall\, t \geq 0,\ -d \;\leq\; v'(t) \;\leq\; +d\,.$$

Therefore the meta-system governing the evolution of the state and the regulon are described by

$$\left\{\begin{array}{lll} (i) & x'(t) \;=\; rx(t)\left(1-\dfrac{x(t)}{b}\right)-v(t)x(t) \\ (ii) & |v'(t)| \;\leq\; d \end{array}\right. \tag{2.12}$$

and the meta-constrained set is the graph of V.

The viability kernel is equal to

$$\mathrm{Viab}(\mathrm{Graph}(V)) \;=\; \{(x,v)\in \mathrm{Graph}(V)\mid x\geq \rho^{\sharp}(v)\}\,,$$

where $\rho^{\sharp}$ is the solution to the differential equation

$$-d\frac{d\rho^{\sharp}}{dv} \;=\; r\left(1-\frac{\rho^{\sharp}(v)}{b}\right)-u\rho^{\sharp}(v)$$

satisfying the initial condition $\rho^{\sharp}(v_{-})=x_{-}$. This viability kernel can also be computed by the Saint-Pierre viability kernel algorithm, as it is shown in Fig. 2.5a.

In this case, danger happens when the economic effort v is larger than v_{-}. One cannot maintain the economic effort constant. The *heavy evolution* consists in keeping the economic effort constant as long as the biomass is larger than $\rho^{\sharp}(v)$. At this level, the economic effort has to be drastically reduced with the velocity d, i.e., using $v(t)=-d$, while the biomass decreases until it reaches the level v_{-}.

2.5.4. The Inert-Schaeffer Meta–System

We have chosen the Verhulst feedback $x\mapsto r(1-\frac{x}{b})$ to represent the growth of the resource to be exploited for anchoring our study in history. But we could have chosen instead the inert feedback $x\mapsto\sqrt{\alpha}\sqrt{2\log\left(\frac{b}{x}\right)}$ and study the viability of the interval $[a,b]$ under the system

$$\left\{\begin{array}{lll} (i) & x'(t) \;=\; x(t)\left(\sqrt{\alpha}\sqrt{2\log\left(\dfrac{b}{x(t)}\right)}-v(t)\right) \\ (ii) & v(t) \;\in\; V\big(x(t)\big) \;:=\; \left[\dfrac{C}{\gamma x(t)-c},\overline{v}\right] \end{array}\right. \quad . \tag{2.13}$$

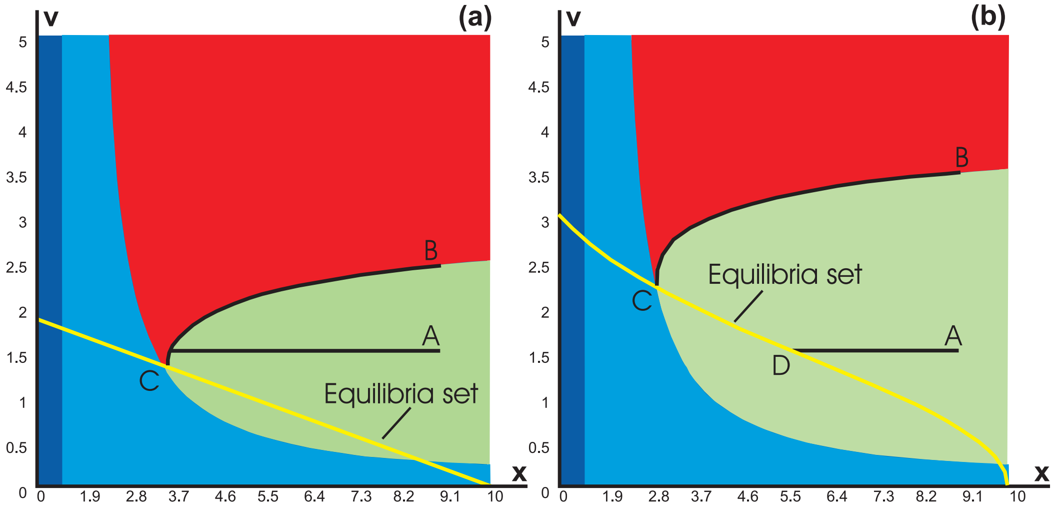

Figure 2.5. Regulation Maps and heavy solutions under Verhulst-Schaeffer (left) and Verhulst-Inert Meta–systems (right) $x'(t) = rx(t)\left(1 - \frac{x(t)}{b}\right) - v(t)\,x(t)$ and $x'(t) = x(t)\left(\sqrt{\alpha}\sqrt{2\log\left(\frac{b}{x(t)}\right)} - v(t)\right)$ respectively. The equilibrium lines are the graphs $r\left(1 - \frac{x}{b}\right)$ and $\sqrt{\alpha}\sqrt{2\log\left(\frac{b}{x}\right)}$. Heavy evolutions stop when their trajectories hit the equilibrium line. (a) The heavy evolutions starting from A or B stop at equilibrium position C. (b) The heavy evolution starting from position A stops at equilibrium position D.

The "equilibrium curve" is here again the union of the singleton $\{0,0\}$ and the graph of the inert feedback $\widetilde{v}(x) = \sqrt{\alpha}\sqrt{2\log\left(\frac{b}{x}\right)}$. They are stable and viable if the graph of $\widetilde{v}(x) := \dfrac{C}{\gamma x - c}$ intersects the equilibrium line: viable equilibrium belong to the interval $[x_-, x_+]$ where $x_\pm$ are the real roots of the equation

$$x = be^{-\frac{1}{2\alpha}\left(\frac{C}{\gamma x - c}\right)^2}$$

Setting $\widetilde{v}(x) = \sqrt{\alpha}\sqrt{2\log\left(\dfrac{b}{x_\pm}\right)}$, we can see that $[a,b]$ is viable whenever $v_- \geq \overline{v}$, i.e., if and only if the inertia bound α is large enough.

We set a bound to the velocity of the economic effort, which translates the rigidity of the economic behavior:

$$\forall\, t \geq 0,\ -d \leq v'(t) \leq +d$$

Therefore the Inert-Schaeffer meta-system governing the evolution of the state and the regulon are described by

$$\left\{ \begin{array}{ll} i) & x'(t) = x(t)\left(\sqrt{\alpha}\sqrt{2\log\left(\dfrac{b}{x(t)}\right)} - v(t)\right) \\ ii) & |v'(t)| \leq d \end{array} \right. \tag{2.14}$$

and the meta-constrained set is the graph of V. The viability kernel can be computed by the viability kernel algorithm, see Fig. 2.5b.

2.5.5. The Crisis Function

We regard $K \subset L$ as a "soft" constrained set embedded in a "hard" constrained set L.

If K is viable, then the evolution may stay in K forever, whereas it has to leave K in finite time if it is a repeller. The question arises whether an evolution reaches the viability kernel of K in finite time, so that it will remain forever in K and, otherwise, if the evolution $x(\cdot)$ reaches K outside its viability kernel, so that the evolution will leave K in finite time and enters a new era of crisis. This crisis may be endless if the evolution enters the complement of the capture basin of K viable in L. Otherwise, same scenario plays again.

Hence the complement in L of the viability kernel of K can be partitioned in two sub–sets, one from which the evolutions will never return to the target (before leaving L), the other one from which at least one evolution returns and remains in the viability kernel of the target after a crisis lasting for a finite time of crisis.

For any $x \in L$, the crisis function, introduced in Doyen and Saint-Pierre (1997), measures the minimal time spent outside the subset K by an evolutions starting at x. In other words, it measures the duration of crisis of not remaining in K. This takes into account the fact that *zero damage within, infinite damage outside the emission corridor*, as it is said in (Tóth 2003), is not the only pre-occupation of viability theory, as well as the tolerable window approach.

It happens that the epigraph of the crisis function is a viability kernel, and can be computed by the viability kernel algorithm, as it is is shown in Fig. 2.6 (for the Verhulst-Schaeffer meta–system) and Fig. 2.8 (for the Inert-Schaeffer meta–system). Figure 2.7 provides the description of the domain of the crisis function and of the projections of the trajectories shown in Fig. 2.6.

2.5.6. Towards Dynamical Games

Actually, since we do not really know what the dynamical equations governing the evolution of the resource are, we could take Malthusian feedbacks $\widetilde{u}$ in a given class $\widetilde{\mathcal{U}}$ of continuous feedbacks as parameters and study the viability kernel $\mathrm{Viab}_{\widetilde{u}}([a,b])$ of the interval $[a,b]$ under the system

$$\begin{cases} (i) & x'(t) \;=\; (\widetilde{u}(x(t)) - v(t))x(t) \\ (ii) & v(t) \;\in\; V(x(t)) \;:=\; \left[\dfrac{C}{\gamma x(t) - c}, \overline{v}\right] \end{cases} \tag{2.15}$$

This suggests to introduce the Guaranteed Viability Kernel

$$\bigcup_{\widetilde{v} \in \widetilde{\mathcal{V}}} \mathrm{Inv}_{\widetilde{v}}([a,b])$$

This is the very first question with which one can study viability issues of dynamical games, that are dynamical systems parameterized by two parameters under the control of two different players. See the paper of Cardaliaguet et al. (1999) for a summary on a viability approach to differential games.

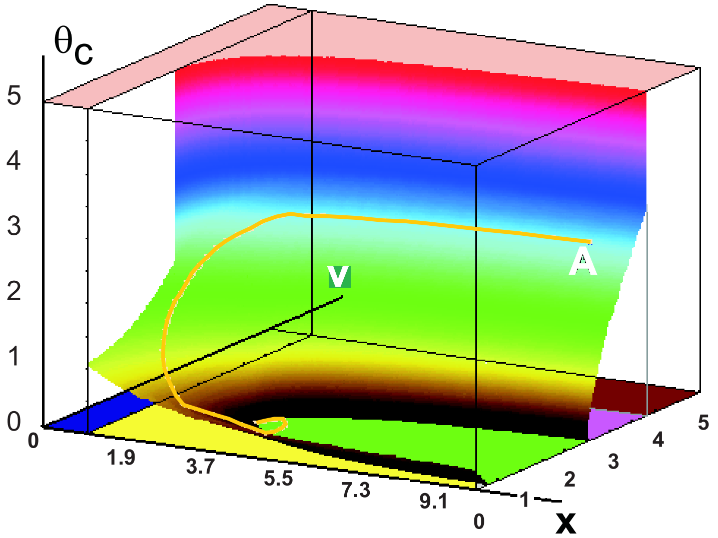

Figure 2.6. Crisis function under the Verhulst-Schaeffer meta–system $x'(t) = rx(t)\left(1-\frac{x(t)}{b}\right) - v(t)x(t)$ modeling the evolution of renewable resources depleted by an economic activity $v(t)$. The meta-controls are the velocities $|v'(t)| \le d$ of economic activity bounded by a constant d. The constrained set $\{(x,v) \in [a,b] \times [0,\overline{v}] \mid v \ge \frac{C}{\gamma x - c}\}$ translates economic constraints. The figure represents the graph of the crisis function, measuring the time spent by an evolution outside of the constrained set equal to zero on the viability kernel (in green), taking infinite values at states from which it is impossible to reach the constrained set (projection of pink area, which is the brown area). It is strictly positive and finite on states defined on the union of the purple and yellow areas. The curve inlayed in the graph of the crisis time function indicates the evolution of this crisis time along the optimal trajectory starting from position A in the space (x, v, θ_C) where $\theta_C(x_A, v_A)$ is the minimal crisis time one can expect when starting from (x_A, v_A).

We can also study "meta–games" by setting bounds c and d on the velocities of the growth rate $u(t)$ and the exploitation effort $v(t)$, regarded as meta–controls, whereas the meta–states of the meta–game are the triples (x,u,v):

$$\begin{cases} (i) & x'(t) = (u(t) - v(t))x(t) \\ (ii) & u'(t) \in B(0,c) \\ (iii) & v'(t) \in B(0,d) \end{cases} \tag{2.16}$$

subjected to the viability constraints

$$u(t) \in \mathbb{R} \text{ and } \frac{C}{\gamma x(t) - c} \le v(t) \le \overline{v}$$

The guaranteed viability kernel of dynamical game (2.16) is computed by the adequate version of the viability kernel algorithm, as it is shown in Fig. 2.9.

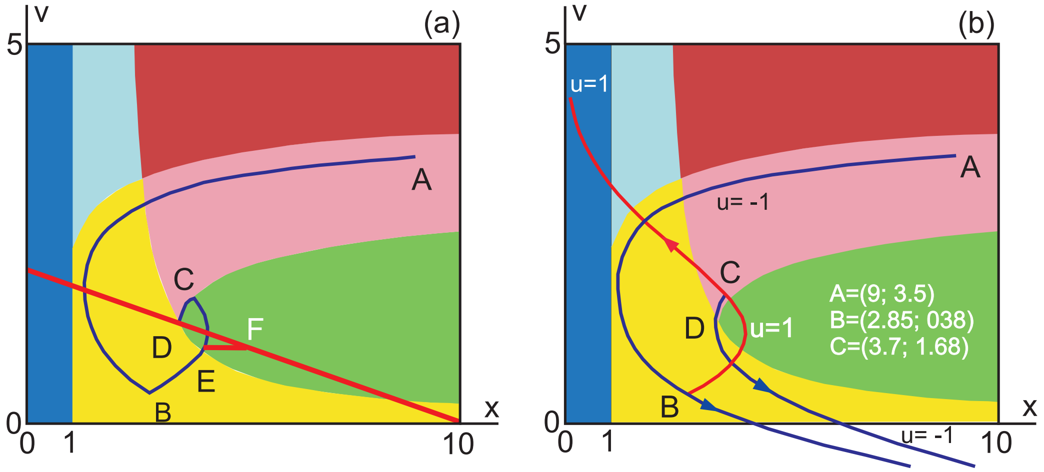

Figure 2.7. Domain of the crisis function $x'(t) = rx(t)\left(1-\frac{x(t)}{b}\right) - v(t)x(t)$ modeling the evolution of renewable resources depleted by an economic activity $v(t)$. The constrained set $\{(x,v)\in[a,b]\times[0,\overline{v}] \mid v \geq \frac{C}{\gamma x-c}\}$ is the epigraph of the hyperbola in these two diagrams. Its viability kernel under Verhulst-Schaeffer meta-system is the green area and the graph of the equilibrium curve is the red line: the viable equilibria range over the intersection of the equilibrium line and of the viability kernel of the constrained set. The projections of the trajectories of heavy and inert evolutions of the crisis function starting from A are shown. As long as they are above the constrained set (purple area), they use the constant meta–control equal to $u=-1$ (slowing down the fishing effort with the maximal (negative) allowed velocity). Being viable, the crisis function remains constant. It starts to decrease when the evolution leaves the constrained set (yellow area) until the time when they reach the state B when the meta-control has to be changed to the constant meta–control equal to $u=+1$ (increasing the fishing effort with the maximal (positive) allowed velocity). Then the evolution reaches the viability kernel at state E. Either we follow the heavy evolution, taking for meta–control the one with minimal velocity. Then the evolution leaves E to reach the equilibrium line in red at state F, where it then remains forever. Or, we keep the meta–control equal to $+1$, which is no longer a viable meta–control: the evolution remains viable until it reaches the boundary of the viability kernel at state C. In order to survive, we have to choose again the meta–control equal to $u=-1$ which brings the evolution to another equilibrium $\mathsf{D}=x_{-}$. The evolution remains there forever taking for new meta–control 0. Source: Patrick Saint-Pierre.

2.6. Viability and Optimality

Interestingly enough, viability theory implies the dynamical programming approach for optimal control. Denote by $\mathcal{S}(x)$ the set of pairs $(x(\cdot),u(\cdot))$ solutions to the control problem given in Eq. (2.4) starting from x at time 0. We consider the minimization problem

$$\begin{cases} V(T,x) = \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}(x)} \inf_{t\in[0,T]} \\ \left(\mathbf{c}(T-t,x(t)) + \int_0^t \mathbf{l}(x(\tau),u(\tau))d\tau\right), \end{cases}$$

where $\mathbf{c}$ and $\mathbf{l}$ are cost functions. We can prove that the graph of the value function $(T,x)\mapsto V(T,x)$ of this optimal control problem is the capture basin of the graph of the cost function $\mathbf{c}$ under an auxiliary system involving (f,U) and the cost function $\mathbf{l}$. The regulation map of this auxiliary system provides the optimal solutions. The tangential conditions furnish

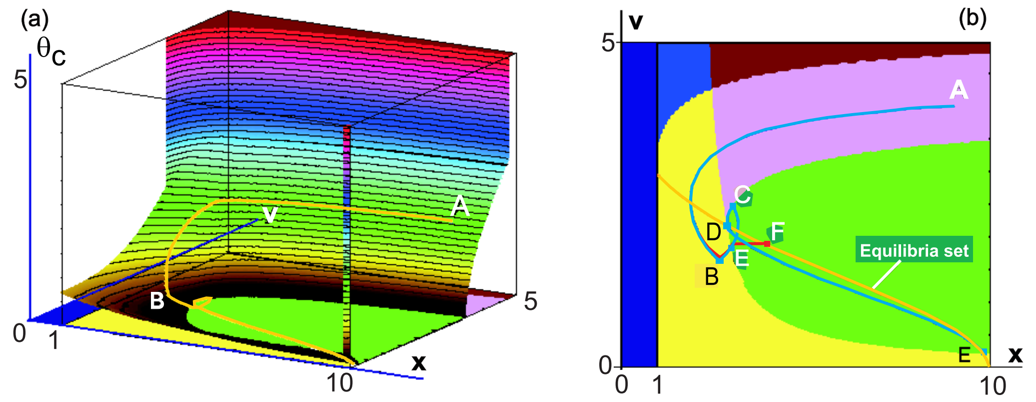

Figure 2.8. Crisis function under the Verhulst-Inert meta–system $x'(t) = x(t)\left(\sqrt{\alpha}\sqrt{2\log\left(\frac{b}{x(t)}\right)} - v(t)\right)$ modeling the evolution of renewable resources depleted by an economic activity $v(t)$. The meta–controls are the velocities $|v'(t)| \leq d$ of economic activity bounded by a constant d. The constrained set $\{(x,v) \in [a,b] \times [0,\overline{v}]\ \}$ translates economic constraints. (a) represents the graph of the crisis function, equal to zero on the viability kernel (in green), taking infinite values at states from which it is impossible to reach the constrained set. (b) is a projection of (a): the union of the brown and light blue areas is the complement in the constrained set of the domain of the crisis function. It is strictly positive and finite on states defined on the union of the purple and yellow areas. The yellow curve is the trajectory of the inert evolution. (cf. also the comments of Figs. 2.6 and 2.7, A-F corresponds to those in Fig. 2.7) Courtesy of Patrick Saint-Pierre.

Hamilton-Jacobi-Bellman equations of which the value function is the solution (see, for instance, Frankowska 1989b; Frankowska 1989a; Frankoswka 1991; Frankowska 1993, and more recently, Aubin 2001a). This is a very general method covering numerous other dynamic optimization problems.

However, contrary to *optimal control theory*, viability theory does not require any single decision-maker (or actor, or player) to guide the system by optimizing an *inter-temporal* optimality criterion[4]. Furthermore, the choice (even conditional) of the controls is not made *once and for all* at some initial time, but *they can be changed at each instant so as to take into account possible modifications of the environment of the system*, allowing therefore for *adaptation* to viability constraints.

Finally, by not appealing to inter-temporal criteria, *viability theory does not require any knowledge of the future*[5] (even of a stochastic nature.) This is of particular importance when experimentation[6] is not possible or when the phenomenon under study is not periodic.

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[4] The choice of which is open to question even in static models, even when multi-criteria or several decision makers are involved in the model.

[5] Most systems we investigate do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

[6] Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to know the future evolution of the system.

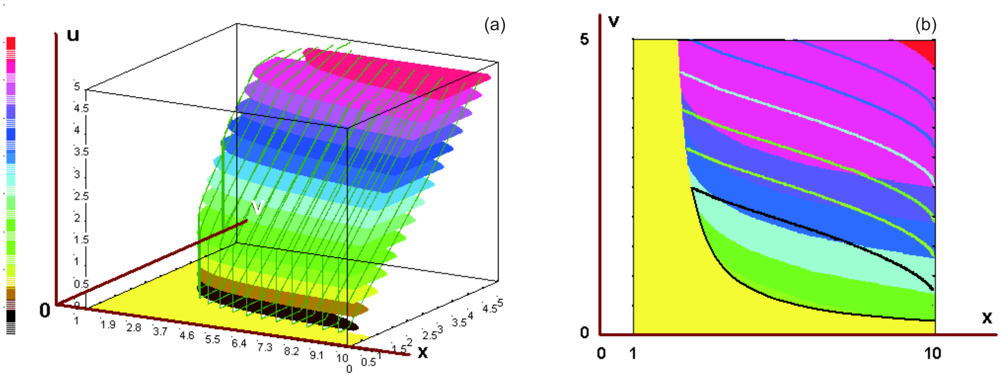

Figure 2.9. Guaranteed viability kernel of the dynamical game (2.16) modeling the evolution of renewable resources $x(t)$ with a growth rate $\|u(t)\|$ depleted by an economic activity $v(t)$. The meta–controls are the velocities $|v'(t)| \leq d$ of economic activity bounded by a constant d and the "meta-tyches" the velocities of the growth rates of the renewable resources. The constrained set $\{(x,v,u) \in [a,b] \times [0,\overline{v}] \times [-c,+c] \mid v \geq \frac{C}{\gamma x - c}\}$ translates economic constraints. The guaranteed viability kernel is represented in the axes (x,v,u). Courtesy of Patrick Saint-Pierre.

For example, in both biological evolution and economics, as well as in other systems we shall investigate, *the dynamics of the system disappear and cannot be recreated*. Hence, *forecasting or prediction of the future are not the issues which we shall address in viability theory.*

However, the conclusions of the theorems allow us to reduce the choice of possible evolutions, or to single out impossible future events, or to provide explanation of some behaviors which do not fit any reasonable optimality criterion. Therefore, instead of using inter-temporal optimization[7] that involves the future, viability theory provides selection procedures of *viable evolutions* obeying, at each instant, state constraints which depend upon the present or the past. (This does not exclude *anticipations*, which are extrapolations of past evolutions, constraining in the last analysis the evolution of the system to be a function of its history.)

2.7. Restoring Viability

The above example shows that there are no reasons why an arbitrary subset K should be viable under a control system. Therefore, the problem of reestablishing viability arises. One can imagine several methods for this purpose:

1. Keep the constraints and change initial dynamics by introducing regulons that are viability multipliers,

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[7] Which can be traced back to Sumerian mythology which is at the origin of Genesis: one decision-maker, deciding what is good and bad and choosing the best (fortunately, on an inter-temporal basis, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week.

2. or change the initial conditions by introducing a *reset map* Φ mapping any state of K to a (possibly empty) set $\Phi(x) \subset X$ of new initialized states, (impulse control),
3. Keep the same dynamics and looking for viable constrained subsets by letting the set of constraints evolve according to mutational equations, as in Aubin (1999b).

We shall describe succinctly these methods.

2.7.1. Designing Regulons

When an arbitrary subset is not viable under an intrinsic system $x'(t) = f\big(x(t)\big)$, the question arises to modify the dynamics by introducing regulons and designing feedbacks so that the constrained subset K becomes viable under the new system. Using the above results and characterizations, one can design several mechanisms. We just highlight three of them, that are described in more details in Aubin (1997).

2.7.1.1. Viability Multipliers

If the constrained set K is of the form

$$K := \{x \in X \quad \text{such that} \ \ h(x) \in M\},$$

where $h : X \mapsto Z := \mathbf{R}^m$ and $M \subset Z$, we regard elements $u \in Z$ as *viability multipliers*, since they play a role analogues to Lagrange multipliers in optimization under constraints. They are candidates to the role of regulons regulating such constraints. Indeed, we can prove that K is viable under the control system

$$x_j'(t) = f_j(x(t)) + \sum_{k=1}^{m} \frac{\partial h_k(x(t))}{\partial x_j} u_k(t)$$

in the same way than the minimization of a function $x \mapsto J(x)$ over a constrained set K is equivalent to the minimization without constraints of the function

$$x \mapsto J(x) + \sum_{k=1}^{m} \frac{\partial h_k(x)}{\partial x_j} u_k$$

for an adequate Lagrange multiplier $u \in Z$ (see for instance Aubin 1998c).

2.7.1.2. Connection Matrices

Instead of introducing viability multipliers, we can use a *connection matrix* $W \in \mathcal{L}(X,X)$ as in neural networks (see, Aubin 1996, for instance). We replace the intrinsic system $x' = \mathbf{I} f(x)$ (where $\mathbf{I}$ denotes the identity) by the system

$$x'(t) \ = \ W(t) \ f\big(x(t)\big)$$

and choose the connection matrices $W(t)$ in such a way that the solutions of the above system are viable in K.

The evolution of the state no longer derives from intrinsic dynamical laws valid in the absence of constraints, but requires some self-organization — described by connection matrices — that evolves together with the state of the system in order to adapt to the viability constraints, the velocity of connection matrices describing the concept of emergence. The evolution law of both the state and the connection matrix results from the confrontation of the intrinsic dynamics to the viability constraints.

One can prove that the regulation by viability multipliers u is a particular case of the regulation by connection matrices W: we associate with x and u the matrix W the matrix the entries of which are equal to

$$w_{i,j} := -\frac{f_i(x)}{\|f(x)\|^2}\sum_{k=1}^{m}\frac{\partial h_k\big(x(t)\big)}{\partial x_j}u_k(t) \text{ if } i \neq j$$

and, of $j = i$,

$$w_{i,i} := 1 - \frac{f_i(x)}{\|f(x)\|^2}\sum_{k=1}^{m}\frac{\partial h_k\big(x(t)\big)}{\partial x_i}u_k(t)\,.$$

The converse is false in general. However, if we introduce the *connectionist complexity index* $\|\mathbf{I} - W\|$, one can prove that *the viable evolutions governed with connection matrices minimizing at each instant the connectionist complexity index are actually governed by the viability multipliers with minimal norms*. See Aubin (1998b), Aubin (1998a), and Aubin (2003).

The concept of heavy evolution when the regulon is a connection matrix amounts to minimize the norm of the velocity $W'(t)$ of the connection matrix $W(t)$ starting from the identity matrix, that can be used as a measure of *dynamical connectionist complexity*. Such a velocity could encapsulate the concept of emergence in the systems theory literature. The connection matrix remains constant — without emergence — as long as the viability of the system is not at stakes, and evolves as slowly as possible.

2.7.1.3. Hierarchical Organization

One can also design dynamic feedbacks for obeying constraints of the form

$$\forall\, t \geq 0,\;\; W^{m-1}(t)\cdots W^{j}(t)\ldots W^{0}(t)x(t) \,\in\, M \,\subset\, Y$$

is satisfied at each instant. Such constraints can be regarded as describing a sequence of m planning procedures.

Introducing at each level of such a hierarchical organization $x_i(t) := W^i(t)x_{i-1}(t)$, one can design dynamical systems modifying the evolution of the intermediate states $x_i(t)$ governed

$$x_j'(t) \;=\; g_j\big(x_j(t)\big)$$

and the entries of the matrices $W^i(t)$ by

$$w_{k,l}^{j'}(t) \;=\; e_{k,l}^{j}\big(W^j(t)\big)\,.$$

Using *viability multipliers*, one can prove that dynamical systems of the form

$$\left\{\begin{array}{lrcl} (1)^0 & x_0'(t) & = & g_0\big(x_0(t)\big) - W^{0^\star}(t)p_1(t)(j=0) \\ (1)^j & x_j'(t) & = & g_j\big(x_j(t)\big) + p_j(t) - W^{j^\star}(t)p_{j+1}(t) \\ & \multicolumn{3}{l}{(j=1,\dots,m-1)} \\ (1)^m & x_m'(t) & = & g_m\big(x_m(t)\big) + p_m(t) \ \ (j=m) \\ (2)^j_{(k,l)} & wj'_{k,l}(t) & = & e^j_{k,l}e\big(W^j(t)\big) - x_{j_k}(t)p_{(j+1)_l}(t) \\ & \multicolumn{3}{l}{(j=0,\dots,m-1,\ k=1,\dots,n,\ l=1,\dots m)} \end{array}\right.$$

govern viable solutions. Here, *the viability multipliers p_j are used as messages to both modify the dynamics of the jth level state $x_j(t)$ and to link to "consecutive levels" $j+1$ and j. Furthermore, the connection matrices evolve in a Hebbian way, since the correction of the velocity $Wj'_{k,l}$ of the entry is the product of the kth component of the j-level intermediate state x_j and the lth component of the $(j+1)$-level viability multiplier p_{j+1}.*

2.7.1.4. Evolution of the Architecture of a Network

This hierarchical organization is a particular case of a network. Indeed, the simplest general form of coordination is to require that a relation between actions of the form $g(A(x_1,\dots,x_n)) \in M$ must be satisfied. Here $A : \prod_{i=1}^n X_i \mapsto Y$ is a *connectionist operator* relating the *individual* actions in a *collective* way. Here $M \subset Y$ is the subset of the *resource space* Y and g is a map, regarded as a propagation map.

We shall study this coordination problem in a *dynamic environment*, by allowing actions $x(t)$ *and connectionist operators* $A(t)$ to evolve according to dynamical systems we shall construct later. In this case, the coordination problem takes the form

$$\forall\, t \geq 0,\ \ g\big(A(t)\big(x_1(t),\dots,x_n(t)\big)\big) \ \in\ M\,.$$

However, in the fields of motivation under investigation, the number n of variables may be very large. Even though the connectionist operators $A(t)$ defining the architecture of the network are allowed to operate *a priori* on all variables $x_i(t)$, they actually operate at each instant t on a *coalition* $S(t) \subset N := \{1,\dots,n\}$ of such variables, varying naturally with time according to the nature of the coordination problem, as in dynamic cooperative games (see for instance Aubin 2005, Scheffran 2001; Scheffran 2001). Therefore, our coordination problem in a dynamic environment involves the evolution

1. of actions $x(t) := \big(x_1(t),\dots,x_n(t)\big) \in \prod_{i=1}^n X_i$,
2. of connectionist operators $A_{S(t)}(t) : \prod_{i=1}^n X_i \mapsto Y$,
3. acting on *coalitions* $S(t) \subset N := \{1,\dots,n\}$ of the n actors

and requires that

$$\forall\, t \geq 0,\ \ g\Big(\big\{A_S(t)\big(x(t)\big)\big\}_{S\subset N}\Big) \ \in\ M\,,$$

where $g : \prod_{S\subset N} Y_S \mapsto Y$. The question we raise is the following. Assume that we may know the intrinsic laws of evolution of the variables x_i (independently of the constraints), of the connectionist operator $A_S(t)$ and of the coalitions $S(t)$, there is no reason why collective constraints defining the above architecture are *viable* under these dynamics, i.e, satisfied at each instant.

One may be able, with a lot of ingeniousness and the intimate knowledge of a given problem, and for simple constraints, to derive dynamics under which the constraints are viable. However, we can investigate whether there is a kind of mathematical factory providing classes of dynamics correcting the initial (intrinsic) ones in such a way that the viability of the constraints is guaranteed. One way to achieve this aim is to use the concept of viability multipliers $q(t)$ ranging over the dual Y^* of the resource space Y that can be used as controls involved for modifying the initial dynamics. This may allow us to provide an *explanation of the formation and the evolution of the architecture of the network and of the active coalitions as well as the evolution of the actions themselves*. In order to tackle mathematically this problem, we shall

1. restrict the connectionist operators to be *multi-affine*, and thus, involve *tensor products*,
2. next, allow coalitions S to become *fuzzy coalitions* so that they can evolve continuously.

Fuzzy coalitions $\chi = (\chi_1, \dots, \chi_n)$ are defined by memberships $\chi_i \in [0,1]$ between 0 and 1, instead of being equal to either 0 or 1 as in the case of usual coalitions. The membership $\gamma_S(\chi) := \prod_{i\in S}\chi_i$ is by definition the product of the memberships of the members $i \in S$ of the coalitions. Using fuzzy coalitions allows us to define their velocities and study their evolution.

The viability multipliers $q(t) \in Y^*$ can be regarded as regulons, i.e., regulation controls or parameters, or virtual prices in the language of economists. They are chosen adequately at each instant in order to show that the viability constraints describing the network can be satisfied at each instant, and the main theorem of this paper guarantees that it is possible. Another one tells us how to choose at each instant such regulons (the regulation law). For each actor i, the velocities $x_i'(t)$ of the state and the velocities $\chi_i'(t)$ of its membership in the fuzzy coalition $\chi(t)$ are corrected by subtracting

1. the sum over all coalitions S to which he belongs of adequate functions *weighted by the membership* $\gamma_S(\chi(t))$:
2. the sum over all coalitions S to which he belongs of the costs of the constraints associated with connectionist tensor A_S of the coalition S weighted by the membership $\gamma_{S\setminus i}(\chi(t))$. This type of dynamics describes a *panurgean effect*. The (algebraic) increase of actor i's membership in the fuzzy coalition aggregates over all coalitions to which he belongs the cost of their constraints weighted by the products of memberships of the actors of the coalition other than him.

As for the correction of the velocities of the connectionist tensors A_S, their correction is a weighted multi-Hebbian" rule: for each component of the connectionist tensor, the correction term is the product of the membership $\gamma S(\chi(t))$ of the coalition S, of the components $x_{i_k}(t)$ and of the component $q^j(t)$ of the regulon. In other words, the viability multipliers appear in the regulation of the multi-affine connectionist operators under the form of *tensor products*, implementing the Hebbian rule for affine constraints (see, Aubin 1996), and multi-Hebbian rules for the multi-affine ones (Aubin and Burnod 1998).

Even though viability multipliers do not provide all the dynamics under which a constrained set is viable, they provide classes of them exhibiting interesting structures that deserve to be investigated and tested in concrete situations.

Remark: Learning Laws and Supply and Demand Law — It is curious that both the standard supply and demand law, known as the "Walrasian tâtonnement process", in economics and the Hebbian learning law in cognitive sciences were the starting points of the Walras general equilibrium theory and neural networks. In both theories, this choice of putting such *adaptation laws* as a prerequisite led to the same "cul de sacs". Starting instead from dynamic laws of agents, viability theory provides dedicated adaptation laws, so to speak, as the conclusion of the theory instead as the primitive feature. In both cases, the point is to maintain the viability of the system, that allocation of scarce commodities satisfy the scarcity constraints in economics, that the viability of the neural network is maintained in the cognitive sciences. For neural networks, this approach provides learning rules that possess the features meeting the Hebbian criterion. For the general networks studied here, these features are still satisfied in spirit. We refer to Aubin (2003) for more details on this topic.

2.7.2. Impulse Systems

There are many other dynamics that obey the inertia principle, among which heavy viable evolutions are the smoothest ones. At the other extreme, one can also study also the (discontinuous) impulsive variations of the regulon. Instead of waiting the system to find a regulon that remains constant for some length of time, as in the case of heavy solutions, one can introduce another (static) system that resets a new constant regulon whenever the viability is at stakes, in such a way that the system evolves until the next time when the viability is again at stakes.

This regulation mode is a particular case of what are called *impulse control* in control theory (see, for instance, Aubin 1999a; Aubin et al. 2002), *hybrid systems* in computer sciences and *integrate and fire* models in neurobiology, etc. Impulse systems are described by a *control system* governing the continuous evolution of the state between two impulsions, and a *reset map* resetting new initial conditions whenever the state enters the domain of the reset map.

An evolution governed by an impulse dynamical system, called a *run* in the control literature, is defined by a sequence of *cadences* (periods between two consecutive impulse times), of *re-initialized states* and of *motives* describing the continuous evolution along a given cadence, the value of a motive at the end of a cadence being reset as the next re-initialized state of the next cadence. Given an impulse system, one can characterize the map providing both the next cadence and the next re-initialized state without computing the impulse system, as a set-valued solution of a system of partial differential inclusions. It provides a summary of the behavior of the impulse system from which one can then reconstitute the evolutions of the continuous part of the run by solving the motives of the run that are the solutions to the dynamical system starting at a given re-initialized state.

A *cadenced run* is defined by *constant cadence, initial state and motive*, where the value at the end of the cadence is reset at the same re-initialized state. It plays the role of discontinuous periodic solutions of a control system. We prove in Aubin and Haddad (2001) that if the sequence of re-initialized states of a run converges to some state, then the run converges to a cadenced run starting from this state, and that, under convexity assumptions, that a cadenced run does exist.

2.7.3. Mutational Equations Governing the Evolution of the Constrained Sets

Alternatively, if the viability constraints can evolve, another way to resolve a viability crisis is to relax the constraints so that the state of the system remains inside the new viability set. For that purpose a kind of differential equation governing the evolution of subsets, called *mutational equations*, have been designed. This requires an adequate definition of the velocity $\overset{\circ}{K}(t)$ of a tube $t \rightsquigarrow K(t)$, called mutation, that makes sense and allows us to prove results analogous to the ones obtained in the domain of differential equations. This can be done, but cannot be described in a few lines. Hence the viability problem amounts to find evolutions of both the state $x(t)$ and the subset $K(t)$ to the system

$$\begin{cases} i) \quad x'(t) = f\big(x(t), K(t)\big) & \text{(differential equation)} \\ ii) \quad \overset{\circ}{K}(t) \ni m\big(x(t), K(t)\big) & \text{(mutational equation)} \end{cases} \tag{2.17}$$

viable in the sense that for every t, $x(t) \in K(t)$. For more details, see Aubin (1999b).

2.8. Conclusion

Viability theory, dealing with the confrontation of uncertain dynamical systems mathematically translated by differential inclusions with constraints and targets has been motivated since the end of the 1970's by an attempt to provide mathematical metaphors of Darwinian evolution and by what appeared as shortcomings of general equilibrium theory in mathematical economics, centered on static considerations dealing with optimal or stationary evolutions. At that time, differential inclusions were mainly studied in Eastern Europe and former Soviet Union (around Filippov, Krasovski, Olech, to quote a few) and viability was restricted to differential equations after the Nagumo Theorem proved in 1943 – and rediscovered at least 14 times since. Since the beginning of the 1980's when the main viability theorems were proved by Haddad (in the framework of differential inclusions with memory, see Haddad 1981), many advances have been accomplished when the concepts of viability kernels and capture basins have been characterized after 1985. It happened that not only they are appealing and natural concepts, but that also they appear as mathematical tools to solve many other problems. We mentioned in this paper the concepts of meta-systems and crisis functions, but more and more mathematical objects happen to be viability kernels or capture basins : Among them, the value functions of many diverse inter-temporal finite or infinite horizon optimal control problems, attractors and fluctuation basins, solutions to first-order systems of partial differential equations or inclusions, issues in qualitative physics, etc., can be characterized through viability kernels and capture basins of auxiliary subsets under auxiliary dynamical systems. Recent advances are being gathered in the Aubin et al. (2006).

If mathematical investigations allow us to derive properties of viability kernels and capture basins, it is quite impossible to characterize by explicit analytical formulas except for very simple examples as the ones we presented here. However, for computational purposes, such formulas are not necessarily needed, and the Saint-Pierre viability kernel algorithm provides not only computations of viability kernels and capture basins, but also viable evolutions such as heavy and inert ones. For the time, as for dynamical programming in optimal control theory, the implementation of this algorithm in computer softwares faces the sadly celebrated dimensional curse. For the time, the general algorithm can be implemented to systems.

Much remains to be done as the theoretical level, the numerical and computing level, and at the modeling level. It is time to cross the interdisciplinary gap and to confront and hopefully to merge the points of view rooted in different disciplines. Mathematics, thanks to its abstraction power by isolating only few key features of a class of problems, can help to bridge these barriers as long as it proposes new methods motivated by these new problems instead of applying the classical ones only motivated until now by physical sciences. If we accept that physics studies much simpler phenomena than the ones investigated by social and biological sciences, and that for this very purpose, they motivated and used a more and more complex mathematical apparatus, we have to accept also that social sciences require a new and dedicated mathematical arsenal which goes beyond what is presently available. Paradoxically, the very fact that the mathematical tools useful for social and biological sciences are and have to be quite sophisticated impairs their acceptance by many social scientists, economists and biologists, and the gap menaces to widen.

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References

Aubin, J.-P. (1991). *Viability Theory*. Boston: Birkhäuser.

Aubin, J.-P. (1996). *Neural Networks and Qualitative Physics: A Viability Approach*. Cambridge: Cambridge University Press.

Aubin, J.-P. (1997). *Dynamic Economic Theory: A Viability Approach*. Berlin: Springer-Verlag.

Aubin, J.-P. (1998a). *Connectionist Complexity and its Evolution*, pp. 50–79. Amsterdam: Elsevier.

Aubin, J.-P. (1998b). *Knowledge and Information in a Dynamic Economy*, Chapter Minimal Complexity and Maximal Decentralization, pp. 83–104. Berlin: Springer-Verlag.

Aubin, J.-P. (1998c). *Optima and Equilibria*. Berlin: Springer-Verlag.

Aubin, J.-P. (1999a). Impulse differential inclusions and hybrid systems: A viability approach. Lecture notes, University of California at Berkeley, Berkeley.

Aubin, J.-P. (1999b). *Mutational and morphological analysis: tools for shape regulation and morphogenesis*. Boston: Birkhäuser.

Aubin, J.-P. (2000). *Applied functional analysis*. New York: Wiley-Interscience.

Aubin, J.-P. (2001a). Concise introduction to viability theory, optimal control and robotics. Technical Report Notes, Ecole Normale Supérieure de Cachan.

Aubin, J.-P. (2001b). Viability kernels and capture basins of sets under differential inclusions. *SIAM J. Control* **40**, 853–881.

Aubin, J.-P. (2002). Boundary-value problems for systems of Hamilton-Jacobi-Bellman inclusions with constraints. *SIAM Journal on Control and Optimization* **41**(2), 425–456.

Aubin, J.-P. (2003). Regulation of the evolution of the architecture of a network by tensors operating on coalitions of actors. *Journal of Evolutionary Economics* **13**, 95–124.

Aubin, J.-P. (2005). Dynamic core of fuzzy dynamical cooperative games. In A. S. Nowak and K. Szajowski (Eds.), *Advances in Dynamic Games: Applications to Economics, Finance, Optimization, and Stochastic Control*, Boston, pp. 129–162. 9th Int. Symp. on Dynamical Games and Applications: Birkhäuser.

Aubin, J.-P., A. Bayen, N. Bonneuil, and P. Saint-Pierre (2006). *Viability, Control and Game Theories: Regulation of Complex Evolutionary Systems Under Uncertainty*. Berlin: Springer-Verlag. to appear.

Aubin, J. P. and Y. Burnod (1998). Hebbian learning in neural networks with gates. Technical Report #981, Université de Paris Dauphine, Cahiers du Centre de Recherche Viabilité, Jeux, Contrôle, Paris.

Aubin, J.-P. and A. Cellina (1984). *Differential Inclusions*. Berlin: Springer-Verlag.

Aubin, J.-P. and G. Da Prato (1998). The viability theorem for stochastic differential inclusions. *Stochastic Analysis and Applications* **16**, 1–15.

Aubin, J.-P. and O. Dordan (1996). *Hung Nguyen*, Chapter Fuzzy Systems, Viability Theory and Toll Sets, pp. 461–488. Dordrecht: Kluwer Academic Publ.

Aubin, J.-P. and O. Dordan (2002). Hybrid systems: Computation and control. In C. J. Tomlin and M. R. Greenstreet (Eds.), *Dynamical Qualitative Analysis of Evolutionary Systems*, Lecture Notes in Computer Science, pp. 62ff. Berlin: Springer-Verlag.

Aubin, J.-P. and H. Doss (2003). Characterization of stochastic viability of any nonsmooth set involving its generalized contingent curvature. *Stochastic Analysis and Applications* **21**(5), 955 – 981.

Aubin, J.-P. and H. Frankowska (1990). *Set-Valued Analysis*. Boston: Birkhäuser.

Aubin, J.-P. and G. Haddad (2001). Cadenced runs of impulse and hybrid control systems. *International Journal of Robust and Nonlinear Control* **11**(5), 401–415.

Aubin, J.-P., J. Lygeros, M. Quincampoix, S. Sastry, and N. Seube (2002). Impulse differential inclusions: A viability approach to hybrid systems. *IEEE Transactions on Automatic Control* **47**, 2–20.

Bruckner, T., G. Petschel-Held, F. Tóth, H. M. Füssel, C. Helm, and M. Leimbach (1999). Climate change decision-support and the tolerable windows approach. *Enviromental Modelling and Assessment* **4**, 217–234.

Cardaliaguet, P., M. Quincampoix, and P. Saint-Pierre (1999). Set-valued numerical analysis for optimal control and differential games. In M. Bardi, T. Parthasarathy, and T. E. S. Raghavan (Eds.), *Stochastic and Differential Games: Theory and Numerical Methods* (Annals of International Society of Dynamic Games ed.), Vol. 4, pp. 177–247. Boston: Birkh äuser.

Dordan, O. (1995). *Analyse qualitative*. Paris: Masson.

Doyen, L. and D. Gabay (1996). Economie des ressources renouvelables et viabilité. *Actes des journèes Vie, Environnement évolution Sociétés*.

Doyen, L. and P. Saint-Pierre (1997). Scale of viability and minimal time of crisis. *Set-Valued Analysis* **5**, 227–246.

Eisenack, K. and G. Petschel-Held (2002). Graph theoretical analysis of qualitative models in sustainability science. In *Working Papers of 16th Workshop on Qualitative Reasoning*, Sitges, pp. 53–60. available from *http://www.upc.es/web/QR2002/Papers/QR2002%20-%20Eisenack.pdf*.

Eisenack, K., J. Scheffran, and J. P. Kropp (2006). Viability analysis of management frameworks for fisheries. *Environmental Modelling and Assessment* **11**(1), 69–79.

Eisenack, K., H. Welsch, and J. P. Kropp (2006). A qualitative dynamical modelling approach to capital accumulation in unregulated fishery. *Journal for Economic Dynamics and Control*. in press, online: DOI 10.1016/j.jedc.2005.08.004.

Eldredge, N. and S. J. Gould (1972). Punctuated equiibria: an alternative to phyletic gradualism. In S. T. J. M. (Ed.), *Models in Paleobiology*, pp. 82–115. San: Francisco: Freeman, Cooper & Company.

Frankoswka, H. (1991). Lower semicontinuous solutions to Hamilton-Jacobi-Bellman equations. In *Proc. 30th Conference on Decision and Control*, Brighton, pp. 265–270.

Frankowska, H. (1989a). Hamilton-Jacobi-equation: viscosity solutions and generalized gradients. *J. of Math. Analysis and Appl.* **141**, 21–26.

Frankowska, H. (1989b). Optimal trajectories associated to a solution of tangent Hamilton-Jacobi-equations. *Applied Mathematics and Optimization* **19**, 291–311.

Frankowska, H. (1993). Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equation. *SIAM J. on Control and Optimization* **31**(1), 257–272.

Gabay, D. (1994). Modeling the articulation between the economy and the environment. In J.-L. Diaz and J.-L. Lions (Eds.), *Environment, economics, and their mathematical models*, pp. 67–86. Paris: Masson.

Graham, M. (1935). Modern theory of exploiting a fishery and applications to North-Sea trawling. *J. Com. Perm. Intern. Exploitation des Mers* **10**, 264–274.

Haddad, G. (1981). Monotone viable trajectories for functional differential inclusions. *J. Diff. Eq.* **42**, 1–24.

Kuipers, B. J. (1994). *Qualitative Reasoning: Modeling and Simulation with Incomplete Knowledge*. Cambridge: MIT Press.

Petschel-Held, G., H. J. Schellnhuber, T. Bruckner, F. Tóth, and K. Hasselmann (1999). The tolerable windows approach: Theoretical and methodological foundations. *Climatic Change* **41**, 303–331.

Quincampoix, M. and P. Saint-Pierre (1998). An algorithm for viability kernels in Hölderian case: Approximation by discrete dynamical systems. *Journal of Mathematical Systems, Estimation, and Control* **8**(1), 17–29.

Rockafellar, R. T. and R. Wets (1997). *Variational Analysis*. Berlin: Springer-Verlag.

Saint-Pierre, P. (1994). Approximation of the viability kernel. *Applied Mathematics & Optimisation* **29**, 187–209.

Schaeffer, M. B. (1954). Some aspects of the dynamics of populations. *Bull. Inter. Amer. Trop Tuna Comm.* **1**, 26–56.

Scheffran, J. (2000). The dynamic interaction between economy and ecology - cooperation, stability and sustainability for a dynamic-game model of resource conflicts. *Mathematics and Computers in Simulation*, 371–380.

Scheffran, J. (2001). Stability and control of value-cost dynamic games. *European Journal of Operations Research* **9**, 197–225.

Tóth, F. L. (2003). Climate policy in light of climate science: The ICLIPS project. *Climatic Change* **56**(1-2), 7–36.