Characterization of Stochastic Viability of any Nonsmooth Set Involving its Generalized Contingent Curvature

Jean-Pierre Aubin[1] **and Halim Doss**[2]

Abstract

Thanks to the Stroock & Varadhan "Support Theorem" and under convenient regularity assumptions, stochastic viability problems are equivalent to invariance problems for control systems (also called tychastic viability), as it has been singled out by Doss in 1977 for instance. By the way, it is in this framework of invariance under control systems that problems of stochastic viability in mathematical finance are studied.

The Invariance Theorem for control systems characterizes invariance through first-order tangential and/or normal conditions whereas the stochastic invariance theorem characterizes invariance under second-order tangential conditions. Doss's Theorem states that these first-order normal conditions are equivalent to second-order normal conditions *that we expect for invariance under stochastic differential equations for smooth subsets.*

We extend this result to any subset by defining in an adequate way the concept of contingent curvature of a set and contingent epi-Hessian of a function, related to the contingent curvature of its epigraph. This allows us to go one step further by characterizing functions the epigraphs of which are invariant under systems of stochastic differential equations. We shall show that they are (generalized) solutions to either a system of first-order Hamilton-Jacobi equations or to an equivalent system of second-order Hamilton-Jacobi equations.

Keywords: stochastic viability, tychastic viability, support theorem, generalized curvature, generalized Hessian, derivatives of set-valued maps.

Let us consider a σ–complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, an increasing family $\widetilde{\mathcal{F}}$ of σ–sub–algebras $\mathcal{F}_t \subset \mathcal{F}$, two finite dimensional vector-spaces $X := \mathbf{R}^m$, $Y := \mathbf{R}^n$ and W a Y-valued Wiener measure.

Let us denote by $\mathbf{I}_\tau(\xi_\tau, \gamma, \sigma)$ the Itô process associated with the drift $\gamma \in L^2_{\widetilde{\mathcal{F}}}(\Omega, X)$

[1] The author thank Hélène Frankowska for pointing an error in an earlier version of the paper. This paper appeared in *Stochastic Analysis and Applications, 25, 951-981*.

[2] Université Paris-Dauphine
Centre de Recherche Viabilité, Jeux, Contrôle
F-75775 Paris cx (16)

and the diffusion $\sigma \in L^2_{\widetilde{\mathcal{F}}}(\Omega, \mathcal{L}(Y,X))$ by

$$\forall\, t \geq \tau,\ \ \mathbf{I}_\tau(\xi_\tau, \gamma, \sigma)(t) \ := \ \xi_\tau + \int_\tau^t \gamma(s)ds + \int_\tau^t \sigma(s)dW(s)$$

An Itô process is a solution to the stochastic differential equation

$$dx \ = \ \gamma(x(t))dt + \sigma(x(t))dW(t)$$

starting at ξ_0 if

$$\xi(t) \ = \ \mathbf{I}_0(\xi_0, \gamma(\xi), \sigma(\xi))(t) \ := \ \xi_0 + \int_0^t \gamma(\xi(s))ds + \int_0^t \sigma(\xi(s))dW(s)$$

We say that a Itô process $\xi(\cdot)$ is viable in K on some interval $[0, T]$ if

$$\forall\, t \in [0,T],\ \text{for almost all}\ \omega \in \Omega,\ \ \xi_\omega(t) \ \in \ K$$

This condition depends only upon the law $\mathbf{L}(\xi(t))$ of $\xi(t)$ because it can be restated

$$\forall\, t \in [0,T],\ \ \mathbf{L}(\xi(t))(K) \ = \ 1$$

Let us set $\sigma^i(x) := \sigma(x)e^i$ so that $\sigma(x)u = \sum_{i=1}^n u_i \sigma^i(x)$. We associate the Itô drift γ and the diffusion σ the Stratonovitch drift $\mathbf{s}(\gamma, \sigma)$ defined by

$$\mathbf{s}(\gamma,\sigma)(x) \ := \ \gamma(x) - \frac{1}{2}\sum_{i=1}^n D\sigma^i(x)\sigma^i(x)$$

Thanks to the Stroock & Varadhan "Support Theorem", we know that, under convenient regularity assumptions on the drift and the diffusion, a closed subset K is invariant under the stochastic differential equation if and only if K is invariant under the tychastic system

$$\left\{ \begin{array}{ll} i) & x'(t) \ = \ \mathbf{s}(\gamma,\sigma)(x(t)) + \sum_{i=1}^n \sigma^i(x(t))v_i \\ ii) & v^i \in \mathbf{R}\ \ i = 1, \ldots, n \end{array} \right.$$

This is very advantageous because the Saint-Pierre Viability Algorithm ([69, Saint-Pierre], [60, Pujal & Saint-Pierre] and [59, Pujal] for instance) allows us to check numerically whether a given subset is invariant under such a "tychastic system" or else, provides the largest subset of K that enjoys this property, called the invariance kernel (see Chapter 5 of [7, Aubin], for instance). We shall not take up this issue in this paper. Rather, we shall characterize this invariance property.

Indeed, when the subset $K := \{x \in X \mid \varphi(x) \ \leq \ 0\}$ is the level set of a smooth function φ such that $\varphi'(x) \neq 0$ whenever $\varphi(x) = 0$, it was proved in [36, Doss] that K is invariant under the above differential inclusion if and only if for every $x \in \partial K = \{x | \varphi(x) = 0\}$,

$$\begin{cases} i) & \langle \varphi'(x), \sigma^i(x) \rangle \ = \ 0, \ \ i = 1, \dots, n \\ ii) & \langle \varphi'(x), \mathbf{s}(\gamma, \sigma)(x) \rangle \ \leq 0 \end{cases}$$

(See also Remark 10 of [37, Doss & Lenglart]). Recently, Halim Doss proved that this condition is equivalent to the second-order condition : for every x satisfying $\varphi(x) = 0$,

$$\begin{cases} i) & \langle \varphi'(x), \sigma^i(x) \rangle \ = \ 0, \ \ i = 1, \dots, n \\ ii) & \langle \varphi'(x), \gamma(x) \rangle + \frac{1}{2} \sum_{i=1}^{n} \varphi''(x)(\sigma^i(x), \sigma^i(x)) \ \leq \ 0 \end{cases}$$

that is usually derived from the Itô formula.

By the way, it is in this framework of invariance under control systems that problems of stochastic viability in mathematical finance are studied ([19, Björk], [37, Doss & Lenglart], [40, Filipovic], [45, Jachimiak], [48, 49, 50, 51, 52, 53, Milian], [75, Tessitore & Zabczyk], [78, 80, Zabczyk]) among many other papers on this topic.

We shall take up this issue in this paper by adapting these results to the case when K is any closed subset by using several concepts of set-valued analysis : contingent cone to any subset, normal cones, contingent derivatives of set-valued maps, contingent epi-derivatives of an extended function, subdifferential, etc. The second order condition involving the Hessian $\varphi''(x)$ will be replaced by the concept of contingent curvature of any set K at x that we shall define, as well as the contingent epi-Hessian of an extended function that allows to give another meaning to generalized solutions to second-order Hamilton-Jacobi equations (different from the concept of subjet used for defining viscosity solutions to second-order Hamilton-Jacobi equations in [26, Crandall, Ishii & Lions]). We shall relate the contingent curvature to the epigraph of an extended function to its contingent epi-Hessian.

There are (too) many ways to define second-order derivatives, as one can see by reading Chapter 13 of [68, Rockafellar & Wets] and its historical notes. We shall see that the Doss' proof in the regular case shows requires naturally the concept of graphical derivative of the subgradient introduced in [5, 6, Aubin] for convex sets[3]. This issue was next taken up in a series of papers, among which we quote only [63, 64, 65, 66, Rockafellar], [56, 57, 58, Poliquin & Rockafellar] and [76, Thibault] in the non convex case.

As a consequence, we shall provide yet another characterization of stochastic invariance of a subset K under a stochastic differential equation, that adds to the ones provided

[3]for studying the derivative of the set-valued map associating with parameters the solutions of the primal and dual convex minimixation problem.

in [10, 11, Aubin & Da Prato], [43, Gautier & Thibault] and [12, Aubin, Da Prato & Frankowska], [29, Da Prato & Frankowska] in terms of stochastic tangent cones, in [15, Bardi & Goatin], [16, Bardi] in terms of second order normal cones and subjets of indicators and in [24, Buckdahn, Peng, Quincampoix & Rainer] in terms of distance functions to the set.

This explosion of tangential and normal characterizations of stochastic viability that are all (more or less) equivalent suggests and asks for a thorough direct study of their equivalence. For instance, in a recent paper [31, **?**, Da Prato & Frankowska], the basic property[4] that we shall derive from the Support Theorem and the Invariance Theorem has been proved directly.

We shall apply these results to characterize an (extended) function $\mathbf{v} : \mathbf{R}_+ \times X \mapsto \mathbf{R} \cup \{+\infty\}$ the epigraph of which is invariant under the system of stochastic differential equations

$$\left\{ \begin{array}{lll} i) & d\tau(t) & = \; dt \\ ii) & d\xi(t) & = \gamma(\xi)dt + \sigma(\xi)dW(t) \\ iii) & d\eta(t) & = \; -(\eta(t)\mu(\xi(t)) + \lambda(\xi(t)))dt + \nu(\xi(t))dW(t) \end{array} \right.$$

We shall show that is a generalized solution to

1. either the system of first-order partial differential equations

$$\left\{ \begin{array}{ll} i) & \dfrac{\partial \mathbf{v}(t,x)}{\partial x}\sigma^i(x) \;=\; \nu^i(x), \quad i = 1, \ldots, n \\ & \\ ii) & \dfrac{\partial \mathbf{v}(t,x)}{\partial t} + \dfrac{\partial \mathbf{v}(t,x)}{\partial x}\gamma(x) + \mu(x)\mathbf{v}(x) + \lambda(x) \\ & \\ & +\dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\left(\langle D(\nu^i(x), \sigma^i(x)\rangle - \dfrac{\partial \mathbf{v}(t,x)}{\partial x}D\sigma^i(x)\sigma^i(x)\right) \;\leq\; 0 \end{array} \right.$$

[4] This is

$$N_K(x) \;\subset\; \ker(\sigma(x)^\star)$$

where $N_K(x)$ is the normal cone to K at x and $\sigma(x)^\star$ the transpose of the diffusion.

2. or the system of second-order partial differential equations

$$\begin{cases} i) & \dfrac{\partial \mathbf{v}(t,x)}{\partial x}\sigma^i(x) \;=\; \nu^i(x), \quad i=1,\ldots,n \\ ii) & \dfrac{\partial \mathbf{v}(t,x)}{\partial t} + \dfrac{\partial \mathbf{v}(t,x)}{\partial x}\gamma(x) + \mu(x)\mathbf{v}(x) + \lambda(x) \\ & +\dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\dfrac{\partial^2 \mathbf{v}(t,x)}{\partial x^2}(\sigma^i(x),\sigma^i(x)) \;\le\; 0 \end{cases}$$

For instance, consider the following classical system of stochastic differential equations

$$\begin{cases} i) & d\tau \;=\; dt \\ ii) & d\xi_0 \;=\; \xi_0\rho_0(\xi)dt \\ iii) & d\xi_1 \;=\; \xi_1\rho_1(\xi)dt + \xi_1\sigma dW(t) \\ iv) & d\eta \;=\; (\xi_0\rho_0(\xi)\widetilde{p}_0(\tau,\xi) + \xi_1\rho_1(\xi)\widetilde{p}_1(\tau,\xi) - \mu(\xi(t))\eta)dt + \xi_1\sigma\widetilde{p}_1(\tau,\xi)dW(t) \end{cases}$$

governing the evolution of the prices x_0 of a nonrisky asset, the price x_1 of the risky asset and the value $y := \widetilde{p}_0(t,x)x_0 + \widetilde{p}_1(t,x)x_1$ of a feedback portfolio $(\widetilde{p}_0(t,x),\widetilde{p}_1(t,x))$.

We infer that any function $\mathbf{v}$ the epigraph of which is stochastic invariant under the above system of stochastic differential equations is a (generalized) solution to the Black-Scholes partial differential inequalities:

$$\frac{\partial \mathbf{v}(t,x)}{\partial t} + \left(\frac{\partial \mathbf{v}(t,x)}{\partial x_0}x_0 + \frac{\partial \mathbf{v}(t,x)}{\partial x_1}x_1(x) + \mu(x) - \mathbf{v}(x)\right)\rho_0(x) + \frac{\partial^2 \mathbf{v}(t,x)}{\partial x_1^2}(x_1\sigma)^2 \;\le\; 0$$

In order to take into account the fact that the solution is only lower semicontinuous, we shall replace the gradient by the (regular) subgradients of the function $\mathbf{v}$ and the Hessian of $\mathbf{v}$ by a contingent epi-Hessian that generalizes it, as we shall see later.

Outline — We devote the first section to the definition of contingent curvature of a set. We recall in the second section the Invariance Theorem for control systems. The Characterization Theorem is proved in the third section. We introduce the concept of contingent epi-Hessian of an extended function in the fourth section and we characterize the functions the epigraph of which is invariant under a system of stochastic differential equations as a (generalized) solution to equivalent systems of first and second-order Hamilton-Jacobi partial differential equations. As a particular case, we relate the invariance of the epigraph of a valuation function to the Black-Scholes partial differential equation in finance. □

1 Contingent Curvature

1.1 Contingent and Normal Cones

Let $(K_n)_{n\in\mathbf{N}}$ be a sequence of subsets of a metric space E. The (Painlevé-Kuratowski) upper limit $\mathrm{Limsup}_{n\to\infty}K_n$ of the sequence K_n is the set of cluster points of sequences $x_n \in K_n$, i.e., of limits of subsequences $x_{n'} \in K_{n'}$.

Let $K \subset X$ be a subset of a normed vector space X and $x \in K$. The (Bouligand-Severi) contingent cone $T_K(x)$ is the set of elements v such that there exists a sequence of elements $h_n > 0$ converging to 0 and a sequence of $v_n \in X$ converging to v satisfying

$$\forall\, n \geq 0,\;\; x + h_n v_n \;\in\; K$$

In other words, *the contingent cone $T_K(x)$ is the upper limit of the subsets* $(K - x)/h$ (regarded as "set differential quotients")

$$T_K(x) \;:=\; \mathrm{Limsup}_{h\to 0+}\frac{K-x}{h}$$

Therefore $T_K(x)$ is always a closed cone of "tangent directions" (which is convex when K is convex or, more generally, when the contingent cone is lower semicontinuous[5] at this point, a vector space when K is a smooth manifold).

The polar cone $N_K(x) := (T_K(x))^-$ is called the *normal cone* to K at x. It is also called the *Bouligand-Severi normal cone*, or the *contingent normal cone*, or also, the *sub-normal cone* and more recently, the *regular normal cone* by R.T. Rockafellar and R. Wets. In this paper[6], only (regular) normals are used, so that we shall drop the adjective "regular".

We recall that the graph of the normal cone $N_K : x \in K \rightsquigarrow N_K(x)$ is closed if and only if the contingent cone map $x \rightsquigarrow T_K(x)$ is lower semicontinuous.

1.2 Contingent Derivatives

Let us consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$. The set-valued map $F^\sharp := {\mathrm{Lim}^\sharp}_{n\to+\infty}F_n$ from X to Y defined by

$$\mathrm{Graph}({\mathrm{Lim}^\sharp}_{n\to+\infty}F_n) \;:=\; \mathrm{Limsup}_{n\to\infty}\mathrm{Graph}(F_n)$$

is called the (graphical) upper limit of the set-valued maps F_n.

Even for single-valued maps, this is a weaker convergence than the pointwise convergence:

[5]See SET-VALUED ANALYSIS, [13, Aubin & Frankowska]. A subset K is said to be sleek at $x \in K$ if $T_K(\cdot)$ is lower semicontinuous at this point. A convex subset K is sleek at each of its points.

[6]We could as well use the smaller cone of proximal normals.

1. If $f_n : X \mapsto Y$ converges pointwise to f, then, for every $x \in X$, $f(x) \in f^\sharp(x)$. If the sequence is equicontinuous, then $f^\sharp(x) = \{f(x)\}$.
2. Let $\Omega \subset \mathbf{R}^n$ be an open subset. If a sequence $f_n \in L^p(\Omega)$ converges to f in $L^p(\Omega)$, then
$$\text{for almost all } x \in \Omega, \ \ f(x) \in f^\sharp(x)$$
3. If a sequence $f_n \in L^p(\Omega)$ converges weakly to f in $L^p(\Omega)$, then
$$\text{for almost all } x \in \Omega, \ \ f(x) \in \overline{\text{co}} f^\sharp(x)$$

Let $F : X \rightsquigarrow Y$ be a set-valued map. We introduce the *differential quotients*

$$u \rightsquigarrow \nabla_h F(x,y)(u) \ := \ \frac{F(x+hu)-y}{h}$$

of a set-valued map $F : X \rightsquigarrow Y$ at $(x,y) \in \text{Graph}(F)$. The contingent derivative $DF(x,y)$ of F at $(x,y) \in \text{Graph}(F)$ is the graphical upper limit of differential quotients:

$$DF(x,y) \ := \ \text{Lim}^\sharp{}_{h \to 0+} \nabla_h F(x,y)$$

In other words, v belongs to $DF(x,y)(u)$ if and only if there exist sequences $h_n \to 0^+$, $u_n \to u$ and $v_n \to v$ such that $\forall n \geq 0, \ \ y + h_n v_n \in F(x + h_n u_n)$.

In particular, if $f : X \mapsto Y$ is a single valued function, we set $Df(x) = Df(x, f(x))$.

We deduce the fundamental formula on the graph of the contingent derivative: The graph of the contingent derivative of a set-valued map is the contingent cone to its graph: for all $(x,y) \in \text{Graph}(F)$,

$$\text{Graph}(DF(x,y)) = T_{\text{Graph}(F)}(x,y)$$

1.3 Contingent Curvature of a Set

Since we wish to characterize the stochastic viability conditions for any closed subset, we need to introduce the concept of contingent curvature of a set K at a point $x \in K$ in a normal direction $p \in N_K(x)$.

This concept involves the contingent derivative $DN_K(x,p)$ of the set-valued map $x \rightsquigarrow N_K(x)$ at a point $(x,p) \in \text{Graph}(N_K)$ of its graph.

By definition of the contingent derivative, $dp \in DN_K(x,p)(dx)$ if and only if $dx \in T_K(x)$ and if there exist $h_n \to 0+$, $dx_n \to dx$ and $dp_n \to dp$ such that

$$\forall \ n \geq 0, \ \ p + h_n dp_n \ \in \ N_K(x + h_n dx_n)$$

This requires that the directions dx belong to the contingent cone $T_K(x)$.

Definition 1.1 *We shall say that the subset K is* curvaceous *at $x \in K$ if*

$$\forall\, x \in K, \forall\, p \in N_K(x), \forall\, u \in \overline{\mathrm{co}}(T_K(x), \quad DN_K(x,p)(u) \neq \emptyset$$

and curvaceous if it is curvaceous at every $x \in K$. In this case, the contingent curvature $\mathrm{Curv}_K(x,p)$ *of K at a point $(x,p) \in \mathrm{Graph}(N_K)$ is defined by*

$$\forall\, u, v \in T_K(x), \quad \mathrm{Curv}_K(x,p)(u,v) := \sup_{dp \in DN_K(x,p)(u)} \langle dp, v\rangle$$

When the subset $K := \{x \in X \mid \varphi(x) \leq 0\}$ is the level set of a twice continuously differentiable function φ such that $\varphi'(x) \neq 0$ for every x such that $\varphi(x) = 0$, then whenever $\varphi(x) = 0$, $N_K(x) := \varphi'(x)\mathbf{R}$ and

$$\forall\, u, v \in \ker \varphi'(x), \quad \mathrm{Curv}_K(x, \varphi'(x))(u,v) := \varphi''(x)(u,v)$$

When the subset K is a closed convex subset, denote by Π_K the best approximation projector onto K. Then if p belongs to the normal cone $N_K(x)$ to K at some $x \in K$, we deduce from the relation $x = \Pi_K(x,p)$ the characterization

$$q \in DN_K(x,p)(u) \iff u \in D\Pi_K(x+p)(u+q)$$

between the contingent derivative of the normal cone N_K and the contingent derivative of the projector Π_K (see [5, 6, Aubin]). This result has been generalized has been generalized in [57, Poliquin & Rockafellar] to so-called "prox-regular" subsets K (see Corollary 13.43 of [68, Rockafellar & Wets] for definitions and more results.)

2 Invariance under Control Systems

Let us consider two Lipschitz maps $f : X \mapsto X$ and $\sigma : X \mapsto \mathcal{L}(Y,X)$ and a Lipschitz set-valued map $U : X \rightsquigarrow Y$. We associate with them the "control" system (f, σ, U) governing the evolutions of the state through

$$\begin{cases} i) & x'(t) = f(x(t)) + \sigma(x(t))u(t) \\ ii) & u(t) \in U(x(t)) \end{cases}$$

Even though this is formally a control system, the terminology is not adequate since the parameters u are no controls, but perturbations, disturbances, parameters that are not under the control of the controller or the decision-maker, but rather under an unknown Nature. We suggest to borrow to Charles Peirce[7] the concept of *tyche*, on of the three

[7] Charles Peirce introduced the concept of *tychastic evolution* in [55, Peirce]. In this paper, Peirce associates with the Greek concept of necessity, *ananke*, the concept of *anancastic evolution*, anticipating the "chance and necessity" framework that has motivated viability theory in the first place. Peirce was a prolific and profound philosopher interested in evolution theory after Darwin and Spencer, and he is in particular the founder of semiotics, term that he coined.

words of classical Greek meaning "chance", and to call in this case the control system as a tychastic system, to stress the fact that tyches $u \in U(x)$ play the role of event $\omega \in \Omega$ in stochastic systems. This analogy is further motivated by the Equivalence Theorem between stochastic viability and tychastic viability. We shall say that a subsets $K \subset X$ is invariant under the tychastic system (f, σ, U) if for every initial state $x \in K$ all (absolutely continuous) solutions $x(\cdot)$ of the tychastic system (f, σ, U) starting at x are locally viable in K in the sense that

$$\exists\, T_{x(\cdot)} > 0 \quad \text{such that} \quad \forall\, t \in [0, T_{x(\cdot)}[, \quad x(t) \in K$$

The Invariance Theorem, a consequence of the Filippov Theorem, states (see for instance Theorem of [7, Aubin]):

Theorem 2.1 *Assume that (f, σ, U) is Lipschitz and that K is closed. Then K is invariant under (f, σ, U) if and only if*

$$\forall\, x \in K,\ \forall\, u \in\ U(x) \quad f(x) + \sigma(x)u \ \in\ \overline{\text{co}}(T_K(x))$$

or, equivalently, in dual form, if and only if

$$\forall\, x \in K,\ \forall\, p \in N_K(x), \quad \langle p, f(x)\rangle + \sup_{u \in U(x)} \langle \sigma(x)^* p, u\rangle \ \leq\ 0 \tag{1}$$

As a consequence, when $U(x) := Y$ is the whole space, we deduce the following

Corollary 2.2 *Assume that the maps f and σ are Lipschitz and that K is closed. Then K is invariant under (f, σ, Y) if and only if*

$$\forall\, x \in K,\ \forall\, p \in N_K(x), \begin{cases} i) & N_K(x) \ \subset\ \ker(\sigma(x)^*) \\ ii) & \langle p, f(x)\rangle \ \leq\ 0 \end{cases} \tag{2}$$

Proof — Indeed, conditions (1) and (2) are obviously equivalent. □

3 Stochastic Dynamical Systems

We associate with the Itô drift γ and the diffusion σ involved in the stochastic differential equation

$$dx \ =\ \gamma(x(t))dt + \sigma(x(t))dW(t)$$

the Stratonovitch drift $\mathbf{s}(\gamma, \sigma)$ defined by

$$\mathbf{s}(\gamma, \sigma)(x) \ :=\ \gamma(x) - \frac{1}{2}\sum_{i=1}^{n} D\sigma^i(x)\sigma^i(x)$$

Theorem 3.1 (Doss) *Let $K \subset X$ be a closed curvaceous subset, $N_K(x)$ the normal cone to K at x and $DN_K(x,p)$ its contingent derivative at $p \in N_K(x)$.*

Let us assume that the drift γ is Lipschitz around K and bounded on K and that the three first derivatives of the diffusions σ^i exist and are bounded on K. The following conditions are equivalent:

1. *K is invariant under the stochastic differential equation $d\xi = \gamma(\xi)dt + \sigma(\xi)dW(t)$,*

2. *K is invariant under the differential inclusion $x'(t) \in \mathbf{s}(\gamma,\sigma)(x(t)) + \mathrm{Im}(\sigma(x(t)))$,*

3. *K satisfies the first-order conditions: for every $x \in K$,*

$$\forall\, p \in N_K(x), \begin{cases} i) & \langle p, \sigma^i(x)\rangle \;=\; 0, \;\; i = 1,\ldots,n \\ ii) & \langle p, \gamma(x)\rangle - \frac{1}{2}\sum_{i=1}^n \langle p, D\sigma^i(x))\sigma^i(x)\rangle \;\leq\; 0 \end{cases}$$

4. *K satisfies the second-order conditions: for every $x \in K$,*

$$\forall\, p \in N_K(x), \begin{cases} i) & \langle p, \sigma^i(x)\rangle \;=\; 0, \;\; i = 1,\ldots,n \\ ii) & \langle p, \gamma(x)\rangle + \frac{1}{2}\sum_{i=1}^n \mathrm{Curv}_K(x,p)(\sigma^i(x),\sigma^i(x)) \;\leq\; 0 \end{cases}$$

Proof — We already mentioned that thanks to the Stroock & Varadhan "Support Theorem", we know that K is invariant under the stochastic differential equation if and only if K is invariant under the tychastic system

$$\begin{cases} i) & x'(t) \;=\; \mathbf{s}(\gamma,\sigma)(x(t)) + \sum_{i=1}^n \sigma^i(x(t))v_i \\ ii) & v^i \in \mathbf{R} \;\; i = 1,\ldots,n \end{cases}$$

Corollary 2.2 (see [36, Doss] in the smooth case) states that K is invariant under the above tychastic system if and only if [8]

$$\forall\, x \in K,\; \forall\, p \in N_K(x), \begin{cases} i) & N_K(x) \;\subset\; \ker(\sigma(x)^*) \\ ii) & \langle p, \mathbf{s}(\gamma,\sigma)(x)\rangle \;\leq 0 \end{cases}$$

Hence the conclusion follows from the following Lemma:

Lemma 3.2 (Doss) *Let $K \subset X$ be a closed curvaceous subset. Let us associate with a drift γ and continuously differentiable diffusions σ^i the Stratonovitch drift $\mathbf{s}(\gamma,\sigma)$*

$$\mathbf{s}(\gamma,\sigma)(x) \;:=\; \gamma(x) - \frac{1}{2}\sum_{i=1}^n D\sigma^i(x)\sigma^i(x)$$

[8] In a recent paper [31, Da Prato & Frankowska], the basic property

$$N_K(x) \;\subset\; \ker(\sigma(x)^*)$$

has been derived directly under weaker assumptions without using the Support Theorem.

Let K be a subset, $N_K(x)$ the normal cone to K at x and $DN_K(x,p)$ its contingent derivative at $p \in N_K(x)$. Let us assume that

$$\forall\, x \in K,\ N_K(x)\ \subset\ \ker(\sigma(x)^*)$$

Then the two following conditions are equivalent:

$$\left\{\begin{array}{ll} i) & \forall\, x \in K,\ \forall\, p \in N_K(x),\ \ \langle p, \mathbf{s}(\gamma,\sigma)(x)\rangle\ \le 0 \\ ii) & \forall\, x \in K,\ \forall\, p \in N_K(x),\ \forall\, dp_i \in DN_K(x,p)(\sigma^i(x)), \\ & \langle p, \gamma(x)\rangle + \frac{1}{2}\sum_{i=1}^n \langle dp_i, \sigma^i(x)\rangle\ \le\ 0 \end{array}\right. \tag{3}$$

Proof — Let us take $dp \in DN_K(x,p)(dx)$. By definition of the contingent derivative, there exist $h_n \to 0+$, $dx_n \to dx$ and $dp_n \to dp$ such that

$$p + h_n dp_n\ \in\ N_K(x + h_n dx_n)$$

Assumption (3.2) can be written

$$\forall\, x \in K,\ \forall\, p \in N_K(x),\ \langle p, \sigma^i(x)\rangle\ =\ 0,\ \ i = 1,\ldots,n$$

Therefore, since σ is continuously differentiable with respect to x, we deduce from assumption (3.2) that for all $\sigma^i(\cdot)$, $\forall\, x \in K$,

$$\left\{\begin{array}{l} 0\ =\ \langle p + h_n dp_n, \sigma^i(x + h_n dx_n)\rangle \\ =\ \langle p, \sigma^i(x)\rangle + h_n(\langle dp_n, \sigma_i(x)\rangle + \langle p, D\sigma^i(x)(dx_n)\rangle) + h_n\varepsilon_n \\ =\ h_n(\langle dp_n, \sigma^i(x)\rangle + \langle p, D\sigma^i(x)(dx_n)\rangle) + h_n\varepsilon_n \end{array}\right.$$

and thus, dividing by $h_n > 0$ and passing to the limit, we obtain that for any $i = 1,\ldots,n$,

$$\forall\, dp \in DN_K(x,p)(dx),\ \ \langle dp, \sigma^i(x)\rangle + \langle p, D\sigma^i(x)dx\rangle\ =\ 0$$

Taking $dx := \sigma^i(x)$, we infer that

$$\forall\, dp_i \in DN_K(x,p)(\sigma^i(x)),\ \ \langle dp_i, \sigma^i(x)\rangle + \langle p, D\sigma^i(x)\sigma^i(x)\rangle\ =\ 0$$

Consequently, we deduce that under assumption (3.2), for any $i = 1,\ldots,n$ and any $dp_i \in DN_K(x,p)(\sigma^i(x))$

$$\left\{\begin{array}{l} \langle p, \mathbf{s}(\gamma,\sigma)(x)\rangle \\ =\ \langle p, \gamma(x)\rangle - \frac{1}{2}\sum_{i=1}^n \langle p, D\sigma^i(x)\sigma^i(x)\rangle \\ =\ \langle p, \gamma(x)\rangle + \frac{1}{2}\sum_{i=1}^n \langle dp_i, \sigma^i(x)\rangle \end{array}\right.$$

and thus, that statements (3)i) and ii) are equivalent. □

4 Contingent Epi-Hessian

4.1 Extended Functions

A function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ is called an *extended (real-valued) function*. Its *domain* is the set of points at which $\mathbf{v}$ is finite:

$$\text{Dom}(\mathbf{v}) \ := \ \{x \in X \ \mid \ \mathbf{v}(x) < +\infty\}$$

A function is said to be *nontrivial* if its domain is not empty. Any function $\mathbf{v}$ defined on a subset $K \subset X$ can be regarded as the extended function $\mathbf{v}_K$ equal to $\mathbf{v}$ on K and to $+\infty$ outside of K, whose domain is K.

Since the order relation on the real numbers is involved in the definition of the Lyapunov property (as well as in minimization problems and other dynamical inequalities), we no longer characterize a real-valued function by its graph, but rather by its *epigraph*

$$\mathcal{E}p(\mathbf{v}) \ := \ \{(x, \lambda) \in X \times \mathbf{R} \mid \mathbf{v}(x) \leq \lambda\}$$

We remark that some properties of a function are actually properties of its epigraph. For instance, *an extended function* $\mathbf{v}$ *is convex (resp. positively homogeneous) if and only if its epigraph is convex (resp. a cone).*

We associate with a subset $\mathcal{M} \subset X \times \mathbf{R}_+$ the function $\mathbf{v}_{\mathcal{M}} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\mathbf{v}_{\mathcal{M}}(x) \ := \ \inf_{(x,w) \in \mathcal{M}} w$$

that we shall call its southern border.

We observe that if $\mathcal{M} \subset X \times \mathbf{R}_+$ is a closed subset, then its southern closure $\mathbf{v}_{\mathcal{M}}$ is lower semicontinuous and

$$\mathcal{E}p(\mathbf{v}_{\mathcal{M}}) \ = \ \mathcal{M} + \{0\} \times \mathbf{R}_+$$

4.2 Contingent Epiderivatives

The epigraph of the *lower epilimit* $\lim^{\sharp}_{\uparrow n \to +\infty} \mathbf{v}_n$ of a sequence of extended functions $\mathbf{v}_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ is the upper limit of the epigraphs:

$$\mathcal{E}p(\lim^{\sharp}_{\uparrow n \to +\infty} \mathbf{v}_n) \ := \ \text{Limsup}_{n \to +\infty} \mathcal{E}p(\mathbf{v}_n)$$

One can check that

$$\lim^{\sharp}_{\uparrow n \to +\infty} \mathbf{v}_n(x_0) \ = \ \liminf_{n \to \infty, x \to x_0} \mathbf{v}_n(x)$$

Let $\mathbf{v} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be a nontrivial extended function and x belong to its domain.

We associate with it the *differential quotients*

$$u \rightsquigarrow \nabla_h \mathbf{v}(x)(u) \ := \ \frac{\mathbf{v}(x+hu)-\mathbf{v}(x)}{h}$$

The contingent epiderivative $D_\uparrow \mathbf{v}(x)$ of $\mathbf{v}$ at $x \in \mathrm{Dom}(\mathbf{v})$ is the lower epilimit of its differential quotients:

$$D_\uparrow \mathbf{v}(x) \ = \ \lim{}^\sharp_{\uparrow h \to 0+} \nabla_h \mathbf{v}(x)$$

One can check that the contingent epiderivative $D_\uparrow \mathbf{v}(x)$ satisfies

$$\forall \, u \in X, \ \ D_\uparrow \mathbf{v}(x)(u) \ = \ \liminf_{h \to 0+, u' \to u} \frac{\mathbf{v}(x+hu')-\mathbf{v}(x)}{h}$$

and the epigraph of the contingent epiderivative $D_\uparrow \mathbf{v}(\cdot)$ is equal to the contingent cone to the epigraph of $\mathbf{v}$ at $(x, \mathbf{v}(x))$

$$\mathcal{E}p(D_\uparrow \mathbf{v}(x)) \ = \ T_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x))$$

Naturally, when $\mathbf{v}$ is Fréchet differentiable at x, then

$$D_\uparrow \mathbf{v}(x)(v) \ = \ \langle f'(x), v \rangle$$

so that *the subdifferential* $\partial_- \mathbf{v}(x)$ *is reduced to the gradient* $\mathbf{v}'(x)$.

The continuous linear functionals $p \in X^\star$ satisfying

$$\forall \, v \in X, \ \ \langle p, v \rangle \ \leq \ D_\uparrow \mathbf{v}(x)(v)$$

are called the *(regular) subgradients* of $\mathbf{v}$ at x, which constitute the (possibly empty) closed convex subset

$$\partial_- \mathbf{v}(x) \ := \ \{ p \in X^\star \mid \forall \, v \in X, \ \ \langle p, v \rangle \ \ \leq \ D_\uparrow \mathbf{v}(x)(v) \}$$

called the *(regular) subdifferential* of $\mathbf{v}$ at x_0.

We also note that

$$\begin{cases} i) & (p,-1) \in N_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x)) \text{ if and only if } p \in \partial_- \mathbf{v}(x) \\ ii) & (p,0) \in N_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x)) \text{ if and only if } p \in \mathrm{Dom}(D_\uparrow \mathbf{v}(x))^- \end{cases}$$

and that

$$N_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x)) \ = \ \{ \lambda(q,-1) \}_{q \in \partial_- \mathbf{v}(x), \, \lambda > 0} \bigcup \{ (q,0) \}_{q \in \mathrm{Dom}(D_\uparrow \mathbf{v}(x))^-}$$

The subset $\text{Dom}(D_\uparrow \mathbf{v}(x))^- = \{0\}$ whenever the domain of the contingent epiderivative $D_\uparrow \mathbf{v}(x)$ is dense in X. This happens when $\mathbf{v}$ is locally Lipschitz and when the dimension of X is finite:

One can also prove that when the dimension of X is finite, the subdifferential $\partial_- \mathbf{v}(x)$ is the set of elements $p \in X^\star$ satisfying

$$\liminf_{x \to x_0} \frac{\mathbf{v}(x) - \mathbf{v}(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \;\geq\; 0 \tag{4}$$

The equivalent formulation (4) of the concept of subdifferential has been introduced in [25, Crandall & P.-L. Lions] for defining *viscosity solutions* to Hamilton-Jacobi equations.

4.3 Contingent Epi-Hessian of an Extended Function

When $K := \mathcal{E}p(\mathbf{v})$ is the epigraph of a nontrivial extended function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$, we recall that the (regular) subdifferential $\partial_- \mathbf{v}(x)$ is the set of elements $p \in X^*$ such that $(p, -1) \in N_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x)) := (T_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x)))^-$.

Definition 4.1 *Let $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended function. Let us consider $x \in \text{Dom}(\partial_- \mathbf{v})$ and $p \in \partial_- \mathbf{v}(x)$.*

We shall denote by $\partial^2_- \mathbf{v}(x,p)$ the set-valued map associating with $dx \in X$ the set of elements $dp \in X^*$ such that there exist sequences $h_n \to 0+$, $u_n \to dx$, $v_n \to D_\uparrow \mathbf{v}(x)(dx)$ and $\pi_n \to dp$ such that

$$\forall\, n \geq 0,\;\; (p + h_n \pi_n, -1) \;\in\; N_{\mathcal{E}p(\mathbf{v})}(x + h_n u_n, \mathbf{v}(x) + h_n v_n)$$

and say that $\mathbf{v}$ is is curvaceous *at $x \in \text{Dom}(\mathbf{v})$ if*

$$\forall\, x \in \text{Dom}(\mathbf{v}), \forall\, p \in \partial_- \mathbf{v}(x), \forall\;\; \partial^2_- \mathbf{v}(x,p) \;\neq\; \emptyset$$

and curvaceous if it is curvaceous at every $x \in \text{Dom}(\mathbf{v})$. In this case, we denote by

$$\text{Hess}_\uparrow(\mathbf{v})(x,p)(u,v) \;:=\; \sup_{dp \in \partial^2_- \mathbf{v}(x,p)(u)} \langle dp, v \rangle$$

the contingent epi-Hessian *of $\mathbf{v}$ at $(x,p) \in \text{Graph}(N_K)$.*

We observe that when the function $\mathbf{v}$ is twice continuously differentiable, then the contingent epi-Hessian of $\mathbf{v}$ at x coincides with its usual Hessian:

$$\text{Hess}_\uparrow(\mathbf{v})(x, \mathbf{v}'(x))(u,v) \;=\; \mathbf{v}''(x)(u,v)$$

We deduce the following

Proposition 4.2 *Let* $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *be a nontrivial curvaceous extended function and* p *belong to the subgradient* $\partial_-\mathbf{v}(x)$*. The two following conditions are equivalent:*

1. $(dp, d\lambda)$ *belongs to* $DN_{\mathcal{E}p(\mathbf{v})}((x, \mathbf{v}(x)), (p, -1))(dx, D_\uparrow \mathbf{v}(x)(dx))$
2. $dp \ \in \ \partial_-^2 \mathbf{v}(x, p)(dx) - p d\lambda$

Therefore, the contingent curvature to the epigraph of a nontrivial extended function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *is related to the contingent epi-Hessian of* $\mathbf{v}$ *by the formula*

$$\left\{ \begin{array}{l} \forall\, p \in \partial_-\mathbf{v}(x), \ \ \mathrm{Curv}_{\mathcal{E}p(\mathbf{v})}((x, \mathbf{v}(x)), (p, -1))((u, D_\uparrow \mathbf{v}(x)(u)), (v, \nu)) \\ = \left\{ \begin{array}{lll} \mathrm{Hess}_\uparrow(\mathbf{v}(x, p))(u, v) & \text{if} & \nu = \langle p, v \rangle \\ +\infty & \text{if not} & \end{array} \right. \end{array} \right.$$

Proof — To say that $(dp, d\lambda)$ belongs to $DN_{\mathcal{E}p(\mathbf{v})}((x, \mathbf{v}(x)), (p, -1))(dx, D_\uparrow \mathbf{v}(x)(dx))$ means that there exist sequences $h_n \to 0+$ and $\varepsilon_{i_n} \to 0, \ (i = 1, 2, 3, 4)$ such that, setting $dy := D_\uparrow \mathbf{v}(x)(dx)$,

$$(p + h_n dp + h_n \varepsilon_{1_n}, -1 + h_n d\lambda + h_n \varepsilon_{2_n}) \ \in \ N_{\mathcal{E}p(\mathbf{v})}(x + h_n dx + h_n \varepsilon_{3_n}, \mathbf{v}(x) + h_n dy + h_n \varepsilon_{4_n})$$

Dividing by $1 - h_n d\lambda - h_n \varepsilon_{2_n} > 0$ for n large enough, setting

$$1 + h_n d\lambda + h_n \varepsilon_{5_n} \ := \ \frac{1}{1 - h_n d\lambda - h_n \varepsilon_{2_n}}$$

and remembering that $N_{\mathcal{E}p(\mathbf{v})}(x)$ is a cone, we infer that

$$((p + h_n dp + h_n \varepsilon_{1_n})(1 + h_n d\lambda + h_n \varepsilon_{5_n}), -1) \ \in \ N_{\mathcal{E}p(\mathbf{v})}(x + h_n dx + h_n \varepsilon_{3_n}, \mathbf{v}(x) + h_n dx + h_n \varepsilon_{4_n})$$

i.e., that $dp + p d\lambda$ belongs to $\partial_-^2 \mathbf{v}(x, p)(dx)$ because $(p + h_n dp + h_n \varepsilon_{1_n})(1 + h_n d\lambda + h_n \varepsilon_{5_n}) = p + h_n(dp + p d\lambda) + h_n \varepsilon_{6_n}$.

Conversely, assume that $p + p d\lambda$ belongs to $\partial_-^2 \mathbf{v}(x, p)$. There exist $h_n \to 0+$ and $\varepsilon_{i_n} \mapsto 0, \ i = 1, 2, 3$ such that

$$(p + h_n(dp + p d\lambda + \varepsilon_{1_n}), -1) \ \in \ N_{\mathcal{E}p(\mathbf{v})}(x + h_n dx + h_n \varepsilon_{2_n}, \mathbf{v}(x) + h_n dy + h_n \varepsilon_{3_n})$$

Dividing by $1 + h_n d\lambda =: \frac{1}{1 - h_n d\lambda + \varepsilon_{4_n}}$, we infer that

$$\left(p + h_n \frac{dp + \varepsilon_{1_n}}{1 + h_n d\lambda}, -1 + h_n d\lambda + \varepsilon_{4_n} \right) \in N_{\mathcal{E}p(\mathbf{v})}(x + h_n dx + h_n \varepsilon_{2_n}, \mathbf{v}(x) + h_n dy + h_n \varepsilon_{3_n})$$

and thus, that $(dp, d\lambda)$ belongs to $DN_{\mathcal{E}p(\mathbf{v})}((x, \mathbf{v}(x)), (p, -1))(dx, D_\uparrow \mathbf{v}(x)(dx))$.

Finally, knowing that $(dp, d\lambda)$ belongs to $DN_{\mathcal{E}p(\mathbf{v})}((x, \mathbf{v}(x)), (p, -1))(dx, dy)$ if and only if $dy = D_\uparrow \mathbf{v}(x)(dx)$ and

$$dp \;=\; d\pi - p d\lambda \text{ where } d\pi \;\in\; \partial_-^2 \mathbf{v}(x, p)(dx)$$

we infer that $\mu := D_\uparrow \mathbf{v}(x)(u)$ and that

$$\begin{cases} \mathrm{Curv}_{\mathcal{E}p(\mathbf{v})}((x, \mathbf{v}(x)), (p, -1))((u, \mu), (v, \nu)) \\ = \; \sup_{(dp, d\lambda) \in DN_{\mathcal{E}p(\mathbf{v})}((x, \mathbf{v}(x)), (p, -1))(u, \mu)} \langle (dp, d\lambda), (v, \nu) \rangle \\ = \; \sup_{d\pi \in \partial_-^2 \mathbf{v}(x, p)(u)} (\langle d\pi, v \rangle) + \sup_{d\lambda \in \mathbf{R}} d\lambda(\nu - \langle p, v \rangle) \end{cases}$$

Therefore

$$\begin{cases} \mathrm{Curv}_{\mathcal{E}p(\mathbf{v})}((x, \mathbf{v}(x)), (p, -1))((u, D_\uparrow \mathbf{v}(x)(u)), (v, \nu)) \\ = \begin{cases} \mathrm{Hess}_\uparrow(\mathbf{v}(x, p))(u, v) & \text{if} \quad \nu = \langle p, v \rangle \\ +\infty & \text{if not} \end{cases} \end{cases}$$

This completes the proof. $\square$

Remark: Epi-Hessian and Derivative of the Subdifferential — One can also relate $\partial_-^2 \mathbf{v}(x, p)$ with the contingent derivative[9] $D\partial_- \mathbf{v}(x, p)$ of the set-valued map $x \rightsquigarrow \partial_- \mathbf{v}(x)$:

Proposition 4.3 *Let $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended function satisfying the regularity assumption:*

$$\forall\, dx \in \mathrm{Dom}(\mathbf{v}),\; \forall\, dx \in \mathrm{Dom}(D_\uparrow \mathbf{v}(x)),\;\; D_\uparrow \mathbf{v}(x)(dx) \;=\; \lim_{h \to 0+,\; u \mapsto dx} \frac{\mathbf{v}(x + hu) - \mathbf{v}(x)}{h}$$

Let p belong to the subgradient $\partial_- \mathbf{v}(x)$.
Then

$$D\partial_- \mathbf{v}(x, p) \;\subset\; \partial_-^2 \mathbf{v}(x, p)$$

Proof — Assume that p belongs to $D\partial_- \mathbf{v}(x, p)(x)$. There exist $h_n \to 0+$ and $\varepsilon_{i_n} \mapsto 0,\; i = 1, 2$ such that

$$p + h_n dp + \varepsilon_{1_n} \;\in\; \partial_- \mathbf{v}(x + h_n dx + h_n \varepsilon_{2_n})$$

or, equivalently, such that

$$(p + h_n dp + h_n \varepsilon_{1_n}, -1) \;\in\; N_{\mathcal{E}p(\mathbf{v})}(x + h_n dx + h_n \varepsilon_{2_n}, \mathbf{v}(x + h_n dx + h_n \varepsilon_{2_n}))$$

[9] This kind of generalized second order derivative has been introduced in [5, 6, Aubin] for the study of the stability of convex optimization problems and thoroughly studied by R.T. Rockafellar and his co-authors in the general case (see Chapter 13 of [68, Rockafellar & Wets]).

Setting $\varepsilon_{3_n} := \frac{\mathbf{v}(x+h_n dx+h_n\varepsilon_{2_n})-\mathbf{v}(x)}{h_n} - D_\uparrow \mathbf{v}(x)(dx)$ that converges to 0 by the regularity assumption, we infer that

$$(p+h_n dp,-1) \in N_{\mathcal{E}p(\mathbf{v})}(x+h_n dx+h_n\varepsilon_{2_n}, \mathbf{v}(x)+h_n D_\uparrow\mathbf{v}(x)(dx)+h_n\varepsilon_{3_n})$$

and thus, that $dp \in \partial_-^2\mathbf{v}(x,p)$. □

5 Stochastically Invariant Epigraphs

Let us consider a system of stochastic differential equations of the form:

$$\begin{cases} i) & d\tau(t) \; = \; dt \\ ii) & d\xi(t) \; = \gamma(\xi)dt + \sigma(\xi)dW(t) \\ iii) & d\eta(t) \; = \; -(\mu(\xi(t))\eta(t) + \lambda(\xi(t)))dt + \nu(\tau(t),\xi(t))dW(t) \end{cases} \tag{5}$$

Let us consider two nontrivial nonnegative lower semicontinuous extended functions $\mathbf{v} : \mathbf{R}_+ \times X \mapsto \mathbf{R} \cup \{+\infty\}$ and $\mathbf{c} : \mathbf{R}_+ \times X \mapsto \mathbf{R} \cup \{+\infty\}$ satisfying

$$\forall\, (t,x) \in \mathbf{R}_+ \times X, \;\; 0 \;\le\; \mathbf{v}(t,x) \;\le\; \mathbf{c}(t,x) \;\le\; +\infty$$

We deduce from Theorem 3.1 and Proposition 4.2 the following characterization of invariant epigraphs:

Theorem 5.1 *Let us assume that the extended functions* $\mathbf{v}$ *and* $\mathbf{c}$ *are lower semicontinuous, curvaceous and nontrivial.*

Let us assume that the drift γ *and the functions* μ *and* λ *are Lipschitz and bounded and that the three first derivatives of the diffusions* σ^i *and* ν^i *exist and are bounded. Then the epigraph of* $\mathbf{v}$ *is stochastically invariant outside the epigraph of of* $\mathbf{c}$ *under the system (5) of stochastic differential equations if and only if* $\mathbf{v}$ *is a solution on*

$$\Omega_{\mathbf{c}}(\mathbf{v}) \;:=\; \{(t,x) \in \mathbf{R}_+ \times X \mid \mathbf{v}(t,x) \;<\; \mathbf{c}(t,x)\}$$

to either one of the following Hamilton-Jacobi partial differential equations:

1. $\mathbf{v}$ *is a solution to the first-order partial differential equation in the sense that: for every* x*,*

$$\begin{cases} \forall\,(p_t,p_x) \in \partial_-\mathbf{v}(t,x), \\ \begin{cases} i) & \langle p_x, \sigma^i(x)\rangle \;=\; \nu^i(t,x), \;\; i=1,\ldots,n \\ ii) & p_t + \langle p_x, \gamma(x)\rangle + \mu(x)\mathbf{v}(x) + \lambda(x) \\ & +\frac{1}{2}\sum_{i=1}^n \left(\langle D_x\nu^i(t,x), \sigma^i(x)\rangle - \langle p_x, D_x\sigma^i(x)\sigma^i(x)\rangle\right) \;\le\; 0 \end{cases} \end{cases}$$

2. $\mathbf{v}$ *is a solution to the second-order partial differential equation in the sense that: for every* x*,*

$$\left\{\begin{array}{l} \forall\,(p_t,p_x)\in\partial_-\mathbf{v}(t,x),\\ \left\{\begin{array}{ll} i) & \langle p_x,\sigma^i(x)\rangle \;=\; \nu^i(t,x),\;\; i=1,\ldots,n\\ ii) & p_t+\langle p_x,\gamma(x)\rangle+\mu(x)\mathbf{v}(x)+\lambda(x)\\ & +\frac{1}{2}\sum_{i=1}^n \mathrm{Hess}_{\uparrow x}(\mathbf{v})(t,x,p_t,p_x)(\sigma^i(x),\sigma^i(x)) \;\leq\; 0 \end{array}\right. \end{array}\right.$$

where we set

$$\mathrm{Hess}_{\uparrow x}(\mathbf{v})(t,x,p_t,p_x)(\sigma^i(x),\sigma^i(x)) \;:=\; \mathrm{Hess}_{\uparrow}(\mathbf{v})((t,x),(p_t,p_x))((0,\sigma^i(x)),(0,\sigma^i(x)))$$

Example: Black and Scholes Partial Differential Equations — To say that an extended function $\mathbf{v}:\mathbf{R}_+\times\mathbf{R}^2$ is a solution to the Black and Scholes partial differential equation amounts to saying that its epigraph $\mathcal{E}p(\mathbf{v})$ is stochastically invariant outside the epigraph of an extended function $\mathbf{c}$ under the system of stochastic differential equations

$$\left\{\begin{array}{lll} i) & d\tau & = \; dt\\ ii) & d\xi_0 & = \; \xi_0\rho_0(\xi)dt\\ iii) & d\xi_1 & = \; \xi_1\rho_1(\xi)dt+\xi_1\sigma dW(t)\\ iv) & d\eta & = \; (\xi_0\rho_0(\xi)\widetilde{p}_0(\tau,\xi)+\xi_1\rho_1(\xi)\widetilde{p}_1(\tau,\xi)-\mu(\xi(t))\eta)dt+\xi_1\sigma\widetilde{p}_1(\tau,\xi)dW(t) \end{array}\right.$$

In other words, the drift $\gamma(x)$ is equal to $(-1,x_0\rho_0(x),x_1\rho_1(x))$, the drift $\lambda(t,x)$ is equal to $-x_0\rho_0(x)\widetilde{p}_0(t,x)-x_1\rho_1(x)\widetilde{p}_1(t,x))$ the diffusion $\sigma(x)$ is equal to $(0,0,x_1\sigma)$ and the diffusion $\nu(t,x)$ to $\widetilde{p}_1(t,x)x_1\sigma$.

Hence the epigraph of $\mathbf{v}$ is stochastic viable outside the epigraph of $\mathbf{c}$ if and only if it is viable under the equivalent "tychastic system"

$$\left\{\begin{array}{lll} i) & t' & = \; 1\\ ii) & x_0' & = \; x_0\rho_0(x)\\ iii) & x_1' & = \; x_1\rho_1(x)-\frac{1}{2}x_1\sigma^2+x_1\sigma u\\ iv) & y' & = \; -\mu(x)y+\widetilde{p}_0(x)x_0\rho_0(x)+\widetilde{p}_1(t,x)x_1\rho_1(x)\\ & & \quad -\dfrac{1}{2}x_1\sigma^2\left(\widetilde{p}_1(t,x)+\dfrac{\partial\widetilde{p}_1(t,x)}{x_1}x_1\right)+\widetilde{p}_1(t,x)x_1\sigma v \end{array}\right.$$

where the tyches (u,v) range over $\mathbf{R}^2$.

We can eliminate the component $\widetilde{p}_0(t,x)$ by writing $\widetilde{p}_0(t,x)x_0 := y-\widetilde{p}_1(t,x)x_1)$ in the equation governing the evolution of of the value of the portfolio: We obtain

$$y' \;=\; (\rho_0(x)-\mu(x))y+\widetilde{p}_1(t,x)x_1(\rho_1(x)-\rho_0(x))-\frac{1}{2}x_1\sigma^2\left(\widetilde{p}_1(t,x)+\frac{\partial\widetilde{p}_1(t,x)}{x_1}x_1\right)+\widetilde{p}_1(t,x)x_1\sigma v$$

We deduce that $\mathbf{v}$ is a solution on

$$\Omega_{\mathbf{c}}(\mathbf{v}) := \{(t,x) \in \mathbf{R}_+ \times X \mid \mathbf{v}(t,x) < \mathbf{c}(t,x)\}$$

to either one of the following Black-Scholes partial differential inequalities:

1. $\mathbf{v}$ is a solution to the first-order partial differential equation in the sense that: for every x, for every $\forall\ (p_t, p_x) \in \partial_-\mathbf{v}(t,x)$,

$$\begin{cases} i) & \widetilde{p}_1(t,x) = p_{x_1} \\ ii) & p_t + p_{x_0}x_0\rho_0(x) - x_0\rho_0(x)\widetilde{p}_0(t,x) + \mu(x)\mathbf{v}(t,x) \\ & +\dfrac{1}{2}\dfrac{\partial \widetilde{p}_1(t,x)}{\partial x_1}x_1^2\sigma^2 \leq 0 \end{cases}$$

2. $\mathbf{v}$ is a solution to the second-order partial differential equation in the sense that for every x, for every $\forall\ (p_t, p_x) \in \partial_-\mathbf{v}(t,x)$,

$$\begin{cases} i) & \widetilde{p}_1(t,x) = p_{x_1} \\ ii) & p_t + (p_{x_0}x_0 + p_{x_1}x_1 + \mu(x) - \mathbf{v}(t,x))\,\rho_0(x) \\ & +\frac{1}{2}\mathrm{Hess}_{\uparrow x_1}(\mathbf{v})(t,x,p_t,p_{x_0},p_{x_1}))(x_1\sigma, x_1\sigma) \leq 0 \end{cases}$$

where we set

$$\mathrm{Hess}_{\uparrow x_1}(\mathbf{v})(t,x,p_t,p_{x_0},p_{x_1}))(x_1\sigma,x_1\sigma) = \mathrm{Hess}_{\uparrow}(\mathbf{v})(t,x,p_t,p_{x_0},p_{x_1}))((0,0,x_1\sigma),(0,0,x_1\sigma))$$

Indeed, we observe that the equation

$$\langle (p_t, p_{x_0}, p_{x_1}), \sigma(x))\rangle = \nu(x)$$

boils down to $p_{x_1}x_1\sigma = \widetilde{p}_1(t,x)x_1\sigma$ and thus, to[10]

$$p_{x_1} = \widetilde{p}_1(t,x)$$

The second-order equation reads

$$\begin{cases} p_t + p_{x_0}x_0\rho_0(x) + p_{x_1}x_1\rho_1(x) + \mu(x)\mathbf{v}(t,x) - \widetilde{p}_0(t,x)x_0\rho_0(x) - \widetilde{p}_1(t,x)x_1\rho_1(x)) \\ +\frac{1}{2}\mathrm{Hess}_{\uparrow x_1}(\mathbf{v})(t,x,p_t,p_{x_0},p_{x_1}))(x_1\sigma,x_1\sigma) \leq 0 \end{cases}$$

[10]Since $\widetilde{p}_1(t,x)$ is regarded as the number of shares of the risky assets, this equation provides the number of shares as the partial derivative of $\mathbf{v}$ with respect to x_1.

Writing that $\widetilde{p}_0(t,x)x_0 := \mathbf{v}(t,x) - \widetilde{p}_1(t,x)x_1 = \mathbf{v}(t,x) - p_{x_1}x_1$, the above equation can be written

$$p_t + (p_{x_0}x_0 + p_{x_1}x_1 + \mu(x) - \mathbf{v}(t,x))\,\rho_0(x) + \frac{1}{2}\mathrm{Hess}_{\uparrow x_1}(\mathbf{v})(t,x,p_t,p_{x_0},p_{x_1}))(x_1\sigma, x_1\sigma) \;\leq\; 0$$

in such a way that the function $\widetilde{p}$ does not appear in it.

The system of first-order partial differential equations reads

$$\begin{cases} i) & \widetilde{p}_1(t,x) \;=\; p_{x_1} \\ ii) & p_t + p_{x_0}x_0\rho_0(x) + p_{x_1}x_1\rho_x(x) - x_0\rho_0(x)\widetilde{p}_0(t,x) - x_1\rho_1(x)\widetilde{p}_1(t,x)) + \mu(x)\mathbf{v}(t,x) \\ & +\frac{1}{2}x_1\sigma^2\left(\frac{\partial\widetilde{p}_1(t,x)}{x_1}x_1 + \widetilde{p}_1(t,x) - p_{x_1}\right) \;\leq\; 0 \end{cases}$$

Therefore, taking into account the first equation, we obtain

$$p_t + \rho_0(x)\,(p_{x_0}x_0 + p_{x_1}x_1 + \mu(x) - \mathbf{v}(t,x)) + \frac{1}{2}\frac{\partial\widetilde{p}_1(t,x)}{\partial x_1}x_1^2\sigma^2 \;\leq\; 0$$

References

[1] AMBROSIO L. (1997) Corso introduttivo alla teoria geometrica della misura ed alle superfici minime, Scuola Normale di Pisa

[2] AMBROSIO L. (1997) Lecture Notes on geometric evolution problems, distance functions and viscosity solutions, Università di Pavia

[3] AMBROSIO L. & MANTEGAZZA C. (to appear) *Curvature and distance function from a manifold*, Scuola Normale di Pisa

[4] AMBROSIO L. & SONER H.M. (1996) *Level-set approach to mean curvature flow in arbitrary codimension*, J. Diff. Geometry, 43, 693-737

[5] AUBIN J.-P. (1982) *Comportement lipschitzien des solutions de problèmes de minimisation convexes*, Comptes-Rendus de l'Académie des Sciences, PARIS, 295, 235-238

[6] AUBIN J.-P. (1984) *Lipschitz behavior of solutions to convex minimization problems*, Mathematics of Operations Research, 8, 87-111

[7] AUBIN J.-P. (1991) Viability Theory Birkhäuser, Boston, Basel

[8] AUBIN J.-P. (2001) *Viability Kernels and Capture Basins of Sets under Differential Inclusions*, SIAM J. Control, 40, 853-881

[9] AUBIN J.-P. & DA PRATO G. (1990) *Stochastic Viability and Invariance*, Annali Scuola Normale di Pisa, 27, 595-694

[10] AUBIN J.-P. & DA PRATO G. (1995) *Stochastic Nagumo's Viability Theorem*, Stochastic Analysis and Applications, 13, 1-11

[11] AUBIN J.-P. & DA PRATO G. (1998) *The Viability Theorem for Stochastic Differential Inclusions*, Stochastic Analysis and Applications, 16, 1-15

[12] AUBIN J.-P., DA PRATO & FRANKOWSKA (2001) *Stochastic Invariance for Differential Inclusions*, Set-Valued Analysis

[13] AUBIN J.-P. & FRANKOWSKA H. (1990) SET-VALUED ANALYSIS, Birkhäuser, Boston, Basel

[14] BARDI M. & CAPUZZO-DOLCETTA I. (1997) OPTIMAL CONTROL AND VISCOSITY SOLUTIONS OF HAMILTON-JACOBI-BELLMAN EQUATIONS, Birkhäuser

[15] BARDI M. & GOATIN P. (1998) *Invariant sets for controlled degenerate diffusions: a viscosity solution approach*, Preprint Padova 2/98

[16] BARDI M. & JENSEN R. (2001) *A geometric characterization of viable sets for controlled degenerate diffusions*

[17] BJORK T. , DI MASI G., KABANOV YU. & RUNGGALDIER W. (to appear) *Towards a general theory of bond markets*, preprint

[18] BJORK T. & SWENSSON L. (1999) *On the existence of finite dimensional realizations for nonlinear forward rate models*, preprint

[19] BJORK T. (1998) ARBITRAGE THEORY IN CONTINUOUS TIME, Oxford University Press

[20] BUCKDAHN R., CARDALIAGUET P. & QUINCAMPOIX M. (to appear) *A representation formula for the mean curvature motion*, SIAM J. Math.Anal.

[21] BUCKDAHN R., QUINCAMPOIX M. & RASCANU A. p h (1997) *Propriétés de viabilité pour des équations différentielles stochastiques rétrogrades et applications à des équations aux dérivées partielles*, Comptes-Rendus de l'Académie des Sciences, 235, 1159-1162

[22] BUCKDAHN R., QUINCAMPOIX M. & RASCANU A. (1998) *Stochastic viability for backward stochastic differential equations and applications to partial differential equations*, Prob. Theory and related fields, 116, 485-504

[23] BUCKDAHN R., QUINCAMPOIX M., RAINER C. & RASCANU A. (to appear) *Stochastic control with exit time and constraints. Applications to small time attainability of sets*, Applied mathematics and Optimization

[24] BUCKDAHN R., PENG S., QUINCAMPOIX M. & RAINER C. (1998) *Existence of stochastic control under state constraints*, Comptes-Rendus de l'Académie des Sciences, 327, 17-22

[25] CRANDALL M.G. & LIONS P.-L. (1983) *Viscosity solutions of Hamilton-Jacobi equations*, Transactions of A.M.S., 277, 1-42

[26] CRANDALL M.G. , ISHII H. & LIONS P.L. (1992) *User's guide to viscosity solutions of second order partial differential equations*, Bull. A:S, 27, 1-67

[27] DA PRATO G. (1995) INTRODUCTION TO DIFFERENTIAL STOCHASTIC EQUATIONS, Appunti dei corsi, Scuola Normale di Pisa

[28] DA PRATO G. (1997) STOCHASTIC EVOLUTION EQUATIONS BY SEMI-GROUP METHODS, Centre de Recerca Matematica

[29] DA PRATO & FRANKOWSKA (1994) *A stochastic Filippov Theorem*, Stochastic Calculus, 12, 409-426

[30] DA PRATO & FRANKOWSKA (to appear) *Invariant measure for a class of parabolic degenerate equations*

[31] DA PRATO & FRANKOWSKA (2001) *Stochastic viability for compact sets in terms of the distance function*, Dynamics Systems Appl., 10, 177-184

[32] DA PRATO & FRANKOWSKA (2004) *Invariance of stochastic control systems with deterministic arguments*, J. Differential Equations, 200. 18-52

[33] DA PRATO G. & ZABCZYK J. (1992) STOCHASTIC EVOLUTION EQUATIONS IN INFINITE DIMENSIONS Cambridge University Press

[34] DELFOUR M. & ZOLESIO J.-P. (2001) SHAPES AND GEOMETRIES: ANALYSIS, DIFFERENTIAL CALCULUS AND OPTIMIZATION, SIAM series on Advances in Design and Control

[35] DELFOUR M. & ZOLESIO J.-P. (to appear) INTRINSIC DIFFERENTIAL GEOMETRY AND THEORY OF THIN SHELLS, Scuola Normale di Pisa

[36] DOSS H. (1977) *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. Henri Poincaré, Calcul des Probabilités et Statistique, 23, 99-125

[37] DOSS H. & LENGLART E. (1977) *Sur le comportement asymptotique des solutions d'équations différentielles stochastiques*, CRAS, 284, 971-974

[38] DOSS H. & PRIOURET P. (1982) *Support d'un processus de réflexion*, Z. Wahrscheinlidchkeitstheorie, 61,, 327-345

[39] EL KAROUI N. (1981) LES ASPECTS PROBABILISTES DU CONTRÔLE STOCHASTIQUE, Springer-Verlag, Lecture Notes in Maths, # 876

[40] FILIPOVIC D. (1999) *Invariant manifolds for weak solutions to stochastic equations*, preprint

[41] FLEMING W.H. & SONER H.M. (1993) *Controlled Markov processes and viscosity solutions*, Springer-Verlag

[42] FRANKOWSKA H. (1993) *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equation*, SIAM J. on Control and Optimization,

[43] GAUTIER S. & THIBAULT L. (1993) *Viability for constrained stochastic differential equations*, Differential Integral Equations, 6, 1395-1414

[44] IKEDA N. & WATANABE S. (1981) STOCHASTIC DIFFERENTIAL EQUATIONS AND DIFFUSION PROCESSES, North-Holland

[45] JACHIMIAK W. (1996) *A note on invariance for semilinear differential equations*, Bull. Pol. Acad. Sc., 44, 179-183

[46] JACHIMIAK W. (1998) *Stochastic invariance in infinite dimension*, preprint

[47] KISIELEWICZ M. (1995) *Viability theorem for stochastic inclusions*, Discussiones Mathematicae, Differential Inclusions, 15, 61-74

[48] MILIAN A. (1993) *A note on the stochastic Invariance for Itô equations*, Bulletin of the Polish Academy of Sciences, 41, 139-150

[49] MILIAN A. (1993) *On multi-dimensional difusion process living in a bounded region*, Stochastic Anal. Appl. 11

[50] MILIAN A. (1994) *Characterization of coefficients of Itô equation whose solutions live in some subsets of* $\mathbf{R}^m$.

[51] MILIAN A. (1995) *Stocastic viability and a comparison theorem*, Colloq. Math.,68,297-316

[52] MILIAN A. (1997) *Invariance for stochastic equations with regular coefficients*, Stochastic Anal. Appl., 15, 91-101

[53] MILIAN A. (1998) *Comparison theorems for stochastic evolution equations*, preprint

[54] MOTYL J. (1999) *viability of set-valued Itô equation*, Bull. Pol. Acad, Sci., Math., 47, 91-103

[55] PEIRCE C. (1893) *Evolutionary love*, The Monist

[56] POLIQUIN R.A. & ROCKAFELLAR R. T. (1994) *Proto-derivative formulas for basic sub-gradient mappings in mathematical programming*, Set-Valued Analysis, 2, 275-290

[57] POLIQUIN R.A. & ROCKAFELLAR R. T. (1996) *Prox-regular functions in variational analysis*, Transactions of the AMS, 348, 1805-1838

[58] POLIQUIN R.A. & ROCKAFELLAR R. T. (1996) *Generalized Hessian properties of regularized nonsmooth functions*, SIAM J. Opt., 6, 1121-1137

[59] PUJAL D. (2000) *Valuation et gestion dynamiques de portefeuilles*, Thèse de l'Université de Paris-Dauphine

[60] PUJAL D. & SAINT-PIERRE P. (2001) *L'algorithme du bassin de capture appliqué pour évaluer des options européennes, américaines ou exotiques*, preprint

[61] ROCKAFELLAR R.T. (1985) *Lipschitzian Properties of Multifunctions*, J. Nonlinear Anal. T.M.A., 9, 867-885

[62] ROCKAFELLAR R.T. (1985) *Maximal Monotone Relations and the Second Derivative of Nonsmooth Functions*, Ann. Inst. H. Poincaré, Analyse Non Linéaire, 2, 167-184

[63] ROCKAFELLAR R.T. (1986) *Lipschitzian stability in optimization: the role of nonsmooth analysis*, IIASA WP-8646

[64] ROCKAFELLAR R.T. (1987) *Pseudo-differentiability of set-valued mappings and its applications in optimization*, (Preprint)

[65] ROCKAFELLAR R.T. (1988) *First and second-order pseudo-differentiability in nonlinear programming*, Trans. AMS., 307, 75-108

[66] ROCKAFELLAR R.T. (1989) *Proto-Differentiability of set-valued mappings and its applications in optimization*, in ANALYSE NON LINÉAIRE, Attouch, Aubin, Clarke, Ekeland Eds., Gauthier-Villars & C.R.M. Université de Montréal, 449-482

[67] ROCKAFELLAR R.T. & URYASEV S. (2000) *Optimizaiton of conditional value-at-risk*, Journal of Risk, 2, 21-41

[68] ROCKAFELLAR R.T. & WETS R. (1997) VARIATIONAL ANALYSIS, Springer-Verlag

[69] SAINT-PIERRE P. (1994) *Approximation of the viability kernel*, Applied Mathematics & Optimisation, 29, 187-209

[70] SONER H.M. & TOUZI N. (1998) *Super-replication under Gamma constraints*, SIAM J. Control and Opt.

[71] SONER H.M. & TOUZI N. (2000) *Dynamic programming for a class of control problems*,

[72] SONER H.M. & TOUZI N. (to appear) *Stochastic target problems, dynamical programming and viscosity solutions*,

[73] STROOCK D.W. & VARADHAN S.R.S. (1972) *On the support of diffusion processes with applications to the strong maximum principle*, Proc. of the sixth Berkeley Symposium on mathematical statistics and Probabilty, Univ. of California Press, Vol. III. Probability Theory, 333-359

[74] STROOCK D.W. & VARADHAN S.R.S. (1979) *Multidimensional diffusion processes*, Springer-Verlag

[75] TESSITORE G. & ZABCZYK J. (1998) *Comments on transition semi-groups and stochastic invariance*, preprint

[76] THIBAULT L. (1983) *Tangent cones and quasi-interiorly tangent cones to multifunctions*, Transactions of the AMS, 277, 601-621

[77] TRUONG X.D.H. (1995) *Existence of viable solutions for nonconvex-valued stochastic differential inclusions in Banach spaces*, Portugalae Mathematica, 52, 241-250

[78] ZABCZYK J. (1996) CHANCE AND DECISION: STOCHASTIC CONTROL IN DISCRETE TIME, Quaderni, Scuola Normale di Pisa

[79] ZABCZYK J. (to appear) *Stable dynamical systems under small perturbations*, Journal of Mathematical Analysis and Applications

[80] ZABCZYK J. (1999) *Stochastic invariance and consistency of financial models*, preprint Scuola Normale di Pisa

Contents