Fuzzy Control Systems and Toll Sets

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Abstract

"Fuzzification" of dynamical and control systems, for which convexity plays a crucial mathematical role, led us to use indicators instead of characteristic functions for imbedding subsets into functions. Therefore, instead of obtaining the genuine fuzzy sets by a convexification procedure, we obtain the "toll sets", the membership function being a cost function taking values between 0 *and* ∞*. With this tool at hands, we can define "fuzzy" — or actually, "toll" — differential inclusions and control systems and solve the viability problem or the toll viability problem by building fuzzy controllers which regulate viable solutions (satisfying at each instant viability and toll viability constraints).*

Furthermore, it happens that the Cramer transform which was introduced to study large deviations maps probability measures to toll sets ! It then reconciles in some way the heated debate between proponents of probabilities and the ones of fuzzy sets — or their clones, toll sets.

Introduction

There are other reasons for introducing and studying fuzzy sets beyond the ones motivating the seminal and pioneering paper[1] by [48, Zadeh] published in 1965.

The one we emphasis here is rooted in the need for convexification procedures which leads us "naturally" to fuzzy sets and toll sets which we shall introduce, study and apply to dynamical and control systems below.

"Fuzzification" of dynamical and control systems, for which convexity plays a crucial mathematical role, led us to use indicators instead of characteristic functions, and thus, the introduction of toll sets.

It happens that the Cramer transform which we introduce below maps probability measures to toll sets ! It then reconciles in some way the heated debate between proponents of probabilities and the ones of fuzzy sets — or their clones, toll sets (see [12, 13, Aubin] and [17, Aubin & Dordan]).

1 Convexification Procedures

Indeed, the first general convexification procedure led to probability measures. Let us explain this on the simple case[2] of a finite set M. We always can embed the discrete set M to $\mathbf{R}^m$ by the "Dirac measure" associating with any $j \in M$ the jth element of the canonical basis of $\mathbf{R}^m$. The convex hull of the image of M by this embedding is the probability simplex of $\mathbf{R}^m$.

Even if one never uses the rich probabilistic interpretation of the convex hull of the set of Dirac measures, this universal embedding of a set without any structure into a convex subset of a vector space is still very useful.

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[1] Since then, it has been wildly successful, above all in many areas outside mathematics. Lately, we found the following quotation by A. Bercoff: *"Aujourd'hui, nous nageons dans la poésie pure des sous ensembles flous" ...*

François Mitterand

in "La lutte finale", Michel Lafon (1994), p.69. Before, the French novelist Jacques Laurent published a novel titled *Les sous-ensembles flous*.

[2] Let X be any set. If we denote by $\mathcal{S}(X)$ the set of functions from a set X to $\mathbf{R}$ supplied with the pointwise convergence topology, one can check that its topological dual $\mathcal{S}^\star(X)$ is the set of finite measures (finite linear combinations of Dirac measures $\delta(x)$: $f \in \mathcal{S}(X) \mapsto f(x) \in \mathbf{R}$). Then the Dirac map transforms points $x \in X$ to Dirac measures $\delta(x) \in \mathcal{S}^\star(X)$, and the convex hull of its image is the set of discrete probability measures. When X is compact, one can check that this set is dense in the closed (actually, weakly compact) convex set of Radon probability measures on X.

Although is convexification procedure is universal, it is not unique. One can for instance use other embeddings taking advantages of added structures of a given set X.

The sets we consider from now on are power spaces. When E is a set, we shall denote by $\mathcal{P}(E)$ or $\mathbf{2}^E$ the shape space or the power space of E, i.e., the family of all subsets of E, including the empty set. For instance, when N is a discrete space with n elements, the shape space $\mathbf{2}^N$ has 2^n elements.

Embedding the power set 2^E into a vector space is quite advantageous, since it allows to exploit the many properties of vector-spaces.

However, we can define several specific embeddings from $\mathbf{2}^E$ to a vector space, each having its own list of advantages and inconveniences. We shall review some of them and use the two first ones of this list for convexification purposes.

1. *Characteristic Map and Fuzzy Sets*

   The characteristic function $\chi_K : E \mapsto \{0,1\}$ associates with $x \in E$ either 1 when $x \in K$ and 0 when $x \notin K$

$$\chi_K(x) := \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if not} \end{cases}$$

   We thus can embed the power set 2^E into the vector space $\mathbf{R}^E$ of real-valued functions defined on E through the map

$$\chi : K \in 2^E \mapsto \chi_K \in \mathbf{R}^E$$

   We observe that the image of the map $\chi : 2^E \mapsto \mathbf{R}^E$ associating with each subset $K \subset E$ its characteristic function χ_K is equal to

$$\chi\left(2^E\right) = \{0,1\}^E$$

   which is the set of the vertices of the hypercube $[0,1]^E$.

   It is then quite natural from a mathematical viewpoint to use the closed convex hull $[0,1]^E$ of the image $\{0,1\}^E$ of the power set 2^E.

   Since we can regard such a characteristic function χ_K as a membership function of an element x to K, by assigning the value 1 whenever x belongs to K and 0 whenever x does not belong to K, and since any element $c \in [0,1]^E$ is a function from E into $[0,1]$, we can interpret it as a membership function of a fuzzy set introduced in 1965 by L. Zadeh and which have been the object of tremendous development.

If a discrete set N contains n elements, the set of fuzzy sets being the convex hull of the power set $\{0,1\}^N$, we can write any fuzzy set in the form

$$c = \sum_{K \in 2^E} m_K \chi_K \text{ where } m_K \geq 0 \ \& \ \sum_{K \in 2^E} m_K = 1$$

The memberships are then equal to

$$\forall\, x \in N, \ \ c(x) \ = \ \sum_{K \ni x} m_K$$

Consequently, if m_K is regarded as the probability for the set K to be formed, the membership of the element x to the fuzzy set c is the sum of the probabilities of the sets containing the element x.

Let $(\Omega, \mathcal{A}, d\mu)$ be a measure space and $\chi : \mathcal{A} :\mapsto L^1(\Omega)$ the map associating with any $K \in \mathcal{A}$ its characteristic function χ_K, which imbeds the σ-algebra $\mathcal{A}$ to the set $L^1(\Omega, \{0,1\})$. We thus can interpret its closed convex hull $L^1(\Omega, [0,1])$ as the set of measurable fuzzy sets. *When the measure μ is nonatomic, C. Castaing derived from the Lyapunov Convexity Theorem that the set $L^1(\Omega, \{0,1\})$ of measurable sets is dense in the set $L^1(\Omega, [0,1])$ of measurable fuzzy sets* (when $L^1(\Omega, \mathbf{R})$ is supplied with the weakened topology).

Remark — The choice of the scale $[0,1]$ is arbitrary. For instance, when N is a set of players in game theory and when subsets $A \subset N$ are regarded as coalitions of players[3], we can, for instance, introduce negative memberships when players enter a coalition with aggressive intents. This is mandatory if one wants to be realistic ! A positive membership is interpreted as a cooperative participation of the player i in the coalition, while a negative membership is interpreted as a non-cooperative participation of the ith player in the generalized coalition.

[3] The idea of using fuzzy coalitions has already been used in the framework of cooperative games with and without side-payments. For instance, it has been shown that in the framework of cooperative games with side payments involving fuzzy coalitions, the concepts of Shapley value and core did coincide with the (generalized) gradient of the "characteristic function" at the "grand coalition". The differences between these concepts for usual games is explained by the different ways one "fuzzyfy" a characteristic function defined on the set of usual coalitions. See [9, 10, Aubin], [8, Aubin, Chapter 12] and [11, Aubin, Chapter 13].

Then, one can replace the cube $[0,1]^n$ by any product $\prod_{i=1}^{n}[\lambda_i, \mu_i]$ for describing the cooperative or noncooperative behavior of the players.

Fuzzy coalitions are also used in dynamical economic theory to explain the evolution of coalitions of economic agents together with the evolution of their consumptions and the evolution of the market prices in order to satisfy the scarcity constraints of each coalitions (see [16, Aubin]). $\square$

2. *Indicator Map and Toll Sets*

Unfortunately, this embedding is not convenient when we want to use the numerous tools of functional analysis, where convexity plays a key role. Indeed, the convexity of a subset of a vector space cannot be characterized by some properties of its characteristic function. But another scaling, amounting to replace characteristic functions by indicators, allows us to characterize convex subsets.

Indeed, any subset $K \subset E$ can be characterized by its *"indicator"* ψ_K, which is the nonnegative extended function defined by:

$$\psi_K(x) \; := \; \begin{cases} 0 & \text{if } \; x \; \in \; K \\ +\infty & \text{if } \; x \; \notin \; K \end{cases}$$

It can be regarded as a "cost function" or a "penalty function", assigning to any element $x \in E$ an infinite cost when x is outside K, and no cost at all when x belongs to K.

We observe that K *is convex (respectively closed) if and only if its indicator is convex (respectively lower semicontinuous).*

The set of indicators being $\{0, \infty\}^E$, its closed convex hull is the set $[0, \infty]^E$ of nonnegative extended functions U from E to $\mathbf{R}_+ \cup \{+\infty\}$. They provide another implementation of the idea underlying "fuzzy sets", in which indicators replace characteristic functions. Instead of using membership functions taking values in the interval $[0, 1]$, we deal with *membership cost functions* taking their values anywhere between 0 and $+\infty$. Following a suggestion of Dubois and Prades, we shall call them "toll sets".

Observe also that in mathematical morphology, toll sets represent gray-scale shapes, i.e., functions associating with each point of the shape its intensity in grey levels. The choice of the representation depends upon the scale. Actually, the real computer scale is $\{0, 256\}$!

3. *Gauge Map*

   Let X be a Banach space. We shall say that the extended function $j_K : x \in X \to j_K(x)$ from X to $[0, \infty]$ defined by

$$j_K(x) \ = \ \inf\{\lambda \text{ such that } \lambda > 0 \text{ and } x \in \lambda K\} \subset [0, \infty]$$

   is the gauge of K, because, when K is a closed convex subset containing the origin, we prove that

$$K \ = \ \{x \in X \mid j_K(x) \ \leq \ 1\}$$

   The gauges satisfy the following properties

$$\left\{ \begin{array}{ll} i) & j_K(\lambda x) \ = \ \lambda j_K(x) \ \text{ for all } \ \lambda \geq 0 \\ & \\ ii) & j_K(x+y) \leq j_K(x) + j_K(y) \end{array} \right. \tag{1}$$

   One observe easily that the gauge of an intersection is the supremum of the gauges.

4. *Support Map*

   Let K be a nonempty subset of a Banach space X. We associate with any linear form $p \in X^\star$

$$\sigma_K(p) \ := \ \sigma(K,p) \ := \ \sup_{x \in K} \langle p, x\rangle \ = \ \sup_{x \in X} (\langle p, x\rangle - \psi_K(x)) \ \in \ \mathbf{R} \cup \{+\infty\}$$

   The function $\sigma_K : X^\star \mapsto \mathbf{R} \cup \{+\infty\}$ is called the support function of K. We say that the subset of $X^\star$ defined by

$$K^- \ := \ \{p \ \in \ X^\star \mid \sigma_K(p) \ \leq \ 0\}$$

   is the (negative) polar cone, of K.

   We observe that

$$\forall \ \lambda, \ \mu > 0, \ \ \sigma_{\lambda L + \mu M}(p) \ = \ \lambda \sigma_L(p) + \mu \sigma_M(p)$$

   and in particular, that if P is a cone, then

$$\sigma_{M+P}(p) \ = \ \left\{ \begin{array}{lll} \sigma_M(p) & \text{if} & p \ \in \ P^- \\ +\infty & \text{if} & p \ \notin \ P^- \ \ \square \end{array} \right.$$

The Separation Theorem can be stated in the following way: *Let K be a nonempty subset of a Banach space X. Its* closed convex hull *is characterized by linear constraint inequalities in the following way:*

$$\overline{co}(K) = \{x \in X \mid \forall\, p \in X^\star, \ \langle p, x\rangle \le \sigma_K(p)\}$$

Furthermore, there is a bijective correspondence between nonempty closed convex subsets of X and nontrivial lower semicontinuous positively homogeneous convex functions on $X^\star$. *Finally, a cone P is closed and convex if and only if $P = P^{--}$ is equal to its bipolar.* (See for instance [11, Aubin] for proofs and more details).

We also observe that *the support function of K is the gauge of the polar set.* We also obtain a general Cauchy-Schwarz-Buniakowski inequality: *Let K be a closed convex containing the origin. Then*

$$\forall\, x \in X, \forall\, p \in X^\star, \ \langle p, x\rangle \le \sigma_K(p)\, j_K(x)$$

5. *Distance Map* We denote by $\mathcal{K}(E)$ the family of nonempty compact subsets of a metric space E.

   We can associate with any subset $K \subset E$ the distance function defined by

$$d_K(x) := d(x, K) := \inf_{y \in K} d(x, y)$$

   We point out that

$$d_{K \cup L} = \min(d_K, d_L)$$

   and that

$$d_K(x) = d_{\overline{K}}(x)$$

   When K is nonempty a closed convex subset, the function d_K is convex.

   We introduce the Hausdorff distance

$$d\!\!l(K, L) := \max(h^\sharp(K, L), h^\sharp(L, K)) \tag{2}$$

   where

$$h^\sharp(K, L) = \sup_{x \in K} d(x, L) = \sup_{x \in K} \inf_{y \in L} d(x, y)$$

and we observe that

$$h^\sharp(K,L) := \sup_{x\in E}(d(x,L) - d(x,K))$$

If E is a compact set, one can check that the map $d : K \in \mathcal{K}(E) \mapsto d(\cdot, K) \in \mathcal{C}(E)$ is an isometry when $\mathcal{K}(E)$ is supplied with the Hausdorff distance and when the space $\mathcal{C}(E)$ of continuous functions is supplied with the uniform convergence topology.

We just provided this nonexhaustive list of embeddings to show that we could incarnate and substantiate Zadeh's idea in many other ways, according to the nature of the problems at hand.

2 Toll Sets

Let X be a finite dimensional vector space. A function $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ is called an extended (real-valued) function. Its domain is the set of points at which V is finite:

$$\mathrm{Dom}(V) \ := \ \{x \in X \ \mid \ V(x) \neq \pm\infty\}$$

A function is said to be nontrivial[4] if its domain is not empty. Any function V defined on a subset $K \subset X$ can be regarded as the extended function V_K equal to V on K and to $+\infty$ outside of K, whose domain is K.

2.1 Toll Sets and Toll Maps

An extended function $V : E \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is characterized by its *epigraph*

$$\mathcal{E}p(V) \ := \ \{(x, \lambda) \ \in \ E \times \mathbf{R} \mid V(x) \ \leq \ \lambda\}$$

An extended function V is convex (resp. positively homogeneous) if and only if its epigraph is convex (resp. a cone.)

We recall that any subset $K \subset X$ can be characterized by its *"indicator"* ψ_K, AND THUS, regarded as a "cost function" or a "penalty function", assigning to any element $x \in X$ an infinite cost when x is outside K, and no cost at all when x belongs to K.

[4]Such a function is said to be proper in convex and non smooth analysis. We chose this terminology for avoiding confusion with proper maps.

We also recall[5] that K *is convex (respectively closed, a cone) if and only if its indicator is convex (respectively lower semicontinuous, positively homogeneous).*

We are led to regard any nonnegative extended function U from X to $\mathbf{R}_+ \cup \{+\infty\}$ as another implementation of the idea underlying "fuzzy sets", in which indicators replace characteristic functions.

Definition 2.1 *We shall regard an extended nonnegative function* $U : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *as a* toll set[6]*. Its* domain *is the domain of* U*, i.e., the set of elements* x *such that* $U(x)$ *is finite, and the* core *of* U *is the set of elements* x *such that* $U(x) = 0$*. The* complement *of the toll set* U *is the complement of its domain and the complement of its core is called the* toll boundary*.*

We shall say that the toll set U *is* convex *(respectively* closed*, a cone) if the extended function* U *is convex (respectively lower semicontinuous, positively homogeneous).*

We observe that the membership function of the empty set is the constant function equal to $+\infty$.

Definition 2.2 *We shall say that a set-valued map* $\mathbf{U} : X \rightsquigarrow Y$ *associating with any* $x \in X$ *a toll subset* $U(x)$ *of* Y *is a* toll set-valued map*. Its* graph *is the toll subset of* $X \times Y$ *associated with the extended nonnegative function* $(x, y) \mapsto U(x, y) := U(x)(y)$ *and its* domain *is*

$$\mathrm{Dom}(\mathbf{U}) \ := \ \{x \in X \mid U(x, y) \ < \ +\infty \text{ for some } y\}$$

A toll set-valued map $\mathbf{U}$ *is said to be* closed *if and only if its graph is closed, i.e., if its membership function is lower semicontinuous. Its values are closed (respectively convex) if and only if the toll subset* $U(x)$ *are closed (respectively convex). It has linear growth if and only if, for some positive constant* c*,*

$$U(x, v) < +\infty \ \Longrightarrow \ \|v\| \leq c(\|x\| + 1)$$

A nontrivial closed toll set-valued map $\mathbf{U}$ *with convex images and linear growth is called a* Marchaud toll set-valued map*.*

[5] Observe that

$$\mathcal{E}p(\psi_K) \ = \ K \otimes \mathbf{R}_+$$

[6] This terminology has been coined by Dubois and Prades.

If $\mathbf{U}$ is a toll set-valued map from X to Y and $\mathbf{V}$ is a toll set-valued map from Y to Z, we define the (composition) product as the toll set-valued map $\mathbf{W} := \mathbf{V} \circ \mathbf{U}$ from X to Z by the cost function

$$W(x,z) \ := \ \inf_{y \in Y} (U(x,y) + V(y,z))$$

and the square product $\mathbf{W}_\square := \mathbf{V} \square \mathbf{U}$ by the cost function

$$W_\square(x,z) \ := \ \sup_{y \in U} (U(x,y) + V(y,z))$$

2.2 Operations on Toll Sets

One can define on toll sets the following operations:

inclusion We shall say that a tall set U_1 is "contained" in a toll set U_2 if and only if their cost functions satisfy $U_1 \geq U_2$.

Intersection of Toll Sets We shall say that the cost function of an "intersection" of toll sets U_i is the sum $\sum_{i \in I} U_i$ of the cost functions.

Minkowski Addition and Difference of Toll Sets Since toll sets are nonnegative extended functions, we can use the epigraphical view point advocated in convex and nonsmooth analysis, which amounts to define operations on functions through operations on their epigraphs.

The closure of the epigraph of the cost function of a toll set is regarded as the epigraph of the cost function of a toll set, called the epiclosure or closure of a toll set.

Let K and B two subsets of a vector space. We set

$$\begin{cases} K + B \ := \ \{y + z\}_{y \in K, z \in B} \ = \ \bigcup_{y \in B} (K + y) \\ \text{(called Minkowski sum of } K \text{ and } B) \\ K - B \ := \ \{y - z\}_{y \in K, z \in B} \ = \ \bigcup_{y \in K} (y - B) \\ \\ \text{(is called the symmetric of } B) \\ K \ominus B \ := \ \{y \mid B + y \subset K\} \ = \ \bigcap_{y \in B} (K - y) \\ \text{(called the Minkowski difference of } K \text{ and } B) \\ \text{(also called the erosion of } K \text{ by } B) \end{cases}$$

The (Minkowski) sum and difference of epigraphs of two cost functions of toll sets V and W are regarded as epigraphs of two cost functions $V \oplus_\uparrow W$ and $V \ominus_\uparrow W$ built from V and W in the following way :

Definition 2.3 *If $V,\ W : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ are two nonempty toll sets, the function $V \oplus_\uparrow W$ defined by*

$$\mathcal{E}p(V \oplus_\uparrow W) \ = \ \mathcal{E}p(V) + \mathcal{E}p(W)$$

is the cost function of the episum *of the toll sets V and W and its epi-closure the* inf-convolution *of the toll sets V and W.*
The cost function $V \ominus_\uparrow W$ defined by

$$\mathcal{E}p(V \ominus_\uparrow W) \ = \ \mathcal{E}p(V) \ominus \mathcal{E}p(W)$$

is the cost function the epidifference *of the toll sets V and W.*

Proposition 2.4 *The epiclosure of the episum $V \oplus_\uparrow W$ is given by the formula*

$$\overline{(V \oplus_\uparrow W)}(x) \ := \ \inf_{y+z=x}(V(y) + W(z)) \ = \ \inf_{y \in X}(V(x-y) + W(y))$$

When the epigraph of V is closed, then

$$(V \ominus_\uparrow W)(x) \ := \ \sup_{y \in X}(V(x+y) - W(y))$$

The epiclosure of the episum is also called the inf-convolution for obvious reasons. This operation appears in several domains of convex analysis[7] and statistics (law of large numbers).

Examples

1. When $W_i := \psi_{K_i}$ are the cost functions of usual sets K_i, the episum of the cost functions

   $$(\psi_{K_1} \oplus_\uparrow \psi_{K_2})(x) \ := \ \psi_{K_1+K_2}(x)$$

   is the cost function of the sum $K_1 + K_2$.
   This is the reason why we regard the episum

   $$(U_1 \oplus_\uparrow U_2)(\cdot)$$

   of the cost functions of two toll sets U_1 and U_2 as the cost function of the "sum" of these two toll sets.

[7] since the *conjugate of the sum of two functions is the inf-convolution of the conjugates.*

2. When $W := \psi_K$ is the cost function of a usual set, the inf-convolution can be written

$$\overline{(V \oplus_{\uparrow} hB)}(x) \;:=\; \inf_{y \in B} V(x - hy)$$

Conjugation Let $V : X \to \mathbf{R} \cup \{+\infty\}$ be the cost function of a toll set. Its conjugate function $V^\star : X^\star \to \mathbf{R} \cup \{+\infty\}$ is defined on the dual of X by

$$\forall\, p \;\in\; X^\star, \;\; V^\star(p) \;:=\; \sup_{x \in X}(< p, x > - V(x))$$

The map $V \mapsto V^\star$ is called the *Fenchel transform*.

For instance, the conjugate of a usual subset K is the support function of K.

We observe the following Fenchel inequality

$$\forall\, x \;\in\; X, \;\; p \;\in\; X^\star, \;\; < p, x > \leq \; V(x) + V^\star(p)$$

One of the two basic theorem of convex analysis states that *a non-trivial extended function $V : X \to \mathbf{R} \cup \{+\infty\}$ is convex and lower semicontinuous if and only if it coincides with its biconjugate.*

In particular, if f U is a toll cone, its conjugate is the cost function of the polar toll cone U^- defined by

$$U^- \;:=\; \{p \in X^\star \mid \forall\, x \in X, \;\; \langle p, x \rangle \leq U(x)\}$$

because

$$U^\star(p) \;=\; \sup_{x \in X}(\langle p, x \rangle - U(x)) \;=\; \psi_{U^-}(p)$$

Furthermore, Moreau and Rockafellar did introduce in the early sixties the concept of subdifferential $\partial V(x)$ of V at x as the subset

$$\partial V(x) \;:=\; \{p \in X^\star \mid \langle p, x \rangle \;=\; V(x) + V^\star(p)\}$$

Therefore, if $V : X \to \mathbf{R} \cup \{+\infty\}$ is convex and lower semicontinuous, *the inverse of the subdifferential $\partial V(\cdot)$ is the subdifferential $\partial V^\star(\cdot)$ of the conjugate function*:

$$p \;\in\; \partial V(x) \;\;\Longleftrightarrow\;\; x \;\in\; \partial V^\star(p)$$

In this case, since $-V^\star(0) \;=\; \inf_{x \in X} V(x)$, then $\partial V^\star(0)$ is the set of minimizers of V, i.e., the set of the "cheapest" elements of the closed convex toll set V.

Remark — As in the case of fuzzy logic, one can construct a "toll logic" (or a fuzzy quantic logic) by representing the set of elements satisfying a given property by a closed convex cone of a finite dimensional vector space. We define the "toll conjunction" as the intersection of toll closed convex cone, the "toll disjunction" as the episum of toll closed convex cone, the "toll negation" through conjugation and the "toll implication" through inclusion of toll sets. □

2.3 The Cramer Transform

There is another (mathematical) reason for which toll sets provide a sensible mathematical representation of the concept of randomness, but different from the representation by probabilities.

The Cramer transform C associates with any nonnegative measure $d\mu$ on a finite dimensional vector space $\mathbf{R}^n$ the nonnegative extended function $C_\mu : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined on $\mathbf{R}^n$ (identified with its dual) by[8] :

$$C_\mu(p) \;:=\; \sup_{x\in\mathbf{R}^n}\left(\langle p,x\rangle - \log\left(\int_{\mathbf{R}^n} e^{\langle x,y\rangle}d\mu(y)\right)\right)$$

Since C_μ is the supremum of affine functions with respect to p, this is a lower semicontinuous convex function. It satisfies

$$C_\mu(p) \;\geq\; \langle p,0\rangle - \log\left(\int_{\mathbf{R}^n} e^{\langle 0,y\rangle}d\mu(y)\right) \;=\; -\log\left(\int_{\mathbf{R}^n} d\mu(y)\right)$$

so that when $d\mu$ is a probability measure, its Cramer transform C_μ is nonnegative and thus, a toll set.

The indicators $\psi_{\{a\}}$ of singleta a are images of *Dirac measures* δ_a: Indeed, if δ_a is the Dirac measure at the point $a \in \mathbf{R}^n$, then

$$\left\{\begin{array}{l} C_{\delta_a}(p) \;=\; \sup_{x\in\mathbf{R}^n}(\langle p,x\rangle - \langle a,x\rangle) \\ = \left\{\begin{array}{ll} 0 & \text{if} \;\; p=a \\ +\infty & \text{if} \;\; p\neq a \end{array}\right. \\ = \;\psi_a(p) \end{array}\right.$$

[8]This Cramer transform plays an important role in statistics, and in particular, in the field of large deviations (see [21, Azencott] for instance). It is the product of the *Laplace transform* $\mu \mapsto \displaystyle\int_{\mathbf{R}^n} e^{\langle x,y\rangle}d\mu(y)$, of the logarithm and of the *Fenchel transform* (conjugate functions) $V(\cdot) \mapsto V^\star(\cdot)$.

The Cramer transform of the Gaussian with mean m and variance σ is the quadratic function $G_{\sigma,\, m}$ defined by

$$G_{\sigma,\, m}(x) \ := \ \frac{1}{2}\left\|\frac{x-m}{\sigma}\right\|^2$$

which we can regard as a Gaussian toll set with mean m and variance σ. Such toll sets play the role of Gaussians in probability theory.

The function $x \mapsto \log\left(\int_{\mathbf{R}^n} e^{\langle x,y\rangle} d\mu(y)\right)$ is

1. *convex*

   Indeed, applying Hölder inequality with exponents $\frac{1}{\alpha_i}$, we obtain

$$\left\{\begin{array}{l}\displaystyle\int_{\mathbf{R}^n} e^{\langle \alpha_1 x_1+\alpha_2 x_2,y\rangle} d\mu(y) \ = \ \int_{\mathbf{R}^n}\left(e^{\langle x_1,y\rangle}\right)^{\alpha_1}\left(e^{\langle x_2,y\rangle}\right)^{\alpha_2} d\mu(y) \\ \displaystyle\leq \left(\int_{\mathbf{R}^n} e^{\langle x_1,y\rangle} d\mu y\right)^{\alpha_1}\left(\int_{\mathbf{R}^n} e^{\langle x_2,y\rangle} d\mu y\right)^{\alpha_2}\end{array}\right.$$

   By taking the logarithms, we get the convexity of this function with respect to x.

2. and *lower semicontinuous*

   Since the measure $d\mu$ is nonnegative, Fatou's Lemma implies that if x_p converges to x, then

$$\int_{\mathbf{R}^n} e^{\langle x,y\rangle} d\mu(y) \ \leq \ \liminf_{p\to\infty}\int_{\mathbf{R}^n} e^{\langle x^p,y\rangle} d\mu(y)$$

   Hence the lower semicontinuity of the Laplace transform of $d\mu$ is established. Since the logarithm is increasing and continuous, it is continuous and nondecreasing.

Therefore

$$C_\mu^\star(x) \ = \ \log\left(\int_{\mathbf{R}^n} e^{\langle x,y\rangle} d\mu(y)\right)$$

It is actually differentiable and its gradient is equal to

$$\nabla C_\mu^\star(x) \ = \ \frac{\int_{\mathbf{R}^n} y e^{\langle x,y\rangle} d\mu(y)}{\int_{\mathbf{R}^n} e^{\langle x,y\rangle} d\mu(y)}$$

When $d\mu$ is the probability law of a random variable, then its mean is equal to $\nabla C_\mu^\star(0)$, which is centered if and only if its Cramer transform vanishes at 0.

Inf-convolution plays the role of the usual convolution product of two integrable functions f and g defined by

$$(f \star g)(x) \ := \ \int_{\mathbf{R}^n} f(x-y)g(y)dy$$

We thus deduce that the Laplace transform of a convolution product is the product of the Laplace transforms because

$$\left\{ \begin{array}{l} \displaystyle\int_{\mathbf{R}^n} e^{\langle x,y\rangle} \int_{\mathbf{R}^n} f(y-z)g(z)dydz \ = \ \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{\langle x,z\rangle} g(z) e^{\langle x,y-z\rangle} g(y-z)dydz \\ \displaystyle = \int_{\mathbf{R}^n} e^{\langle x,z\rangle} g(z)dz \int_{\mathbf{R}^n} e^{\langle x,u\rangle} g(u)du \end{array} \right.$$

Therefore, taking the logarithm, we obtain

$$\left\{ \begin{array}{l} \log\left(\int_{\mathbf{R}^n} e^{\langle x,y\rangle}(f\star g)(y)dy\right) \\ = \ \log\left(\int_{\mathbf{R}^n} e^{\langle x,y\rangle} f(y)dy\right) + \log\left(\int_{\mathbf{R}^n} e^{\langle x,y\rangle} g(y)dy\right) \end{array} \right.$$

Finally, one can prove that if

$$0 \ \in \ \mathrm{Int}\left(\mathrm{Dom}\left(\log\int_{\mathbf{R}^n} e^{\langle \cdot,y\rangle} f(y)dy\right) - \mathrm{Dom}\left(\log\int_{\mathbf{R}^n} e^{\langle \cdot,y\rangle} g(y)dy\right)\right)$$

the Fenchel conjugate of this sum is the inf-convolution of the Fenchel conjugates. This happens for instance when the support of one of these functions is compact. This allows us to conclude that, under this assumption,

$$C_{f\star g} \ = \ C_f \oplus_{\uparrow} C_g$$

In particular, the regularization of a function is obtained by taking its convolution by a Gaussian. The Cramer transform implies that it is the inf-convolution of a function by a quadratic function, called *the Moreau or Moreau-Yosida transform* of a function $V : X \to \mathbf{R} \cup \{+\infty\}$. It is defined by

$$f_\sigma(x) := \inf_{y\in X}\left[f(y) + \frac{1}{2}\|\frac{x-y}{\sigma}\|^2\right] \tag{3}$$

In the same way that the convolution product by a Gaussian maps a function to an infinitely differentiable function, the inf-convolution by a quadratic function maps a lower semicontinuous convex function to a continuously differentiable convex function:

Theorem 2.5 *Let $V : X \to \mathbf{R} \cup \{+\infty\}$ be a nontrivial lower semicontinuous convex function from X to $\mathbf{R} \cup \{+\infty\}$. Then there exists a* unique *solution (denoted by $J_\sigma(x)$) of the minimization problem $V_\sigma(x)$:*

$$V_\sigma(x) = V(J_\sigma x) + \frac{1}{2}\left\|\frac{x - J_\sigma x}{\sigma}\right\|^2.$$

Furthermore, V_σ is convex, continuously differentiable:

$$DV_\sigma(x) \ = \ \frac{x - J_\sigma x}{\sigma^2}$$

When σ converges to 0,

$$\forall x \in \mathrm{Dom}(V), \ \ V_\sigma(x) \to V(x) \ \ \textit{and} \ \ J_\sigma x \to x \tag{4}$$

and when $\sigma \to \infty$,

$$V_\sigma(x) \ \ \textit{converges to} \quad - V^*(0) = \inf_{x \in X} V(x) \tag{5}$$

The Moreau-Yosida transform of an indicator ψ_K is the function

$$x \mapsto \frac{1}{\sigma^2} d(x, K)$$

The quadratic functions

$$G_{\sigma,\, m}(x) \ := \ \frac{1}{2}\left\|\frac{x - m}{\sigma}\right\|^2$$

are regarded as Gaussian toll sets with mean m and variance σ. They form a class stable by inf-convolution:

Proposition 2.6 *The Gaussian toll sets are stable under inf-convolution:*

$$\left(G_{\sigma_1,\, m_1} \oplus G_{\sigma_2,\, m_2}\right)(x) \ = \ G_{\sqrt{\sigma_1^2 + \sigma_2^2},\, m_1 + m_2}$$

Proof — One must compute the solution to the minimization problem

$$\inf_y \left(\frac{1}{2}\left\|\frac{x - y - m_1}{\sigma_1}\right\|^2 + \frac{1}{2}\left\|\frac{y - m_2}{\sigma_2}\right\|^2 \right)$$

From Fermat's Rule, this problem achieves its minimum at

$$\bar{y} \ := \ \frac{\sigma_2^2(x - m_1) + \sigma_1^2 m_2}{\sigma_1^2 + \sigma_2^2}$$

so that

$$\begin{aligned} \bar{y} - (x - m_1) &= \frac{\sigma_1^2(m_1 + m_2 - x)}{\sigma_1^2 + \sigma_2^2} \\ \bar{y} - m_2 &= -\frac{\sigma_2^2(m_1 + m_2 - x)}{\sigma_1^2 + \sigma_2^2} \end{aligned}$$

Consequently,

$$\begin{cases} \left(G_{\sigma_1,\, m_1} \oplus G_{\sigma_2,\, m_2}\right)(x) = \frac{1}{2}\left\|\frac{x - \bar{y} - m_1}{\sigma_1}\right\|^2 + \frac{1}{2}\left\|\frac{\bar{y} - m_2}{\sigma_2}\right\|^2 \\ = \frac{1}{2}\left\|\frac{x - (m_1 + m_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right\|^2 = G_{\sqrt{\sigma_1^2+\sigma_2^2},\, m_1+m_2} \quad \square \end{cases}$$

2.4 Analogy between Integration and Optimization

Beyond the Cramer transform which is only defined for functions or measures defined on vector spaces $\mathbf{R}^n$, there exists an striking formal analogy between optimization and probability theory that we shall only sketch without entering details which may lead us too far.

To the integral

$$\int_\Omega x(\omega) d\mu(\omega)$$

of a nonnegative measurable function defined on a measure space $(\Omega, \mathcal{A}, d\mu)$ corresponds the *infimum on a closed toll set U of a lower semicontinuous function $V : E \mapsto \mathbf{R} \cup \{+\infty\}$ on a metric space E* defined by

$$\inf_{x \in E} (V(x) + U(x))$$

To the Dirac measure $\delta_a : x(\cdot) \mapsto x(a)$ corresponds the indicator ψ_a because

$$\inf_{x \in V} (V(x) + \psi_a(x)) = V(a)$$

To the integral $\int_A d\mu(\omega)$ of the characteristic function of a measurable set $A \in \mathcal{A}$ providing the measure $d\mu$ of a subset A corresponds the minimization problem of a function $V(\cdot)$ on the closed subset A

$$\inf_{x \in E} (\psi_A(x) + V(x)) = \inf_{x \in A} V(x)$$

Consequently, to the measure $d\mu$, which is a function from the σ-algebra $\mathcal{A}$ to the half-line $\mathbf{R}_+$ supplied with the operations $+$ and $\times$, corresponds the

Maslov measure M_V, function from the family of closed subsets of E defined by

$$K \mapsto \inf_{x \in K} V(x)$$

which enjoys the following properties :

$$\begin{cases} i) & M_V(E) \;=\; \inf_{x \in E} V(x) \\ ii) & M_V(\emptyset) \;=\; +\infty \\ iii) & M_V(A \cup B) \;=\; \inf(M_V(A), M_V(B)) \end{cases}$$

The analogy then becomes algebraic, because $(\mathbf{R}_+, +, \times)$ supplied with the usual addition and multiplication on one hand, and $(\mathbf{R}_+, \inf, +)$ supplied with the infimum and the usual addition on the other hand, are two instances of "dioids, which are kind of rings supplied with two operations which do not have inverses: They are defined as follows:

Definition 2.7 *A* commutative dioid *is a set D supplied with*

1. *an associative, commutative and idempotent operation $\vee$, the neutral element of which is denoted by $\emptyset$*
2. *an associative commutative operation $\oplus$, the neutral element of which is denoted by $\{0\}$*

satisfying the distributivity condition

$$\begin{cases} i) & \forall\, x, y, z \in \mathcal{D}, \;\; (x \vee y) \oplus z \;=\; (x \oplus z) \vee (y \oplus z) \\ ii) & \forall\, x \in \mathcal{D}, \;\; x \oplus \emptyset \;=\; \emptyset \end{cases}$$

Besides the dioids $(\mathbf{R}_+, +, \times)$ and $(\mathbf{R}_+, \inf, +)$ (where the neutral element of the infimum is $\{+\infty\}$), another dioid is famous in mathematical morphology, a branch of image processing: it is the dioid of the family of subsets of a finite dimensional vector space supplied with the union of sets and the (Minkowski) addition of sets.

3 Fuzzy or Toll Differential Inclusions

3.1 Set-Valued Maps

Let X and Y be two spaces. A set-valued map F from X to Y is characterized by its *graph Graph*(F), the subset of the product space $X \times Y$ defined by

$$\text{Graph}(F) \;:=\; \{(x, y) \in X \times Y \;|\; y \in F(x)\}$$

We shall say that $F(x)$ is the *image* or the *value* of F at x.

A set-valued map is said to be *nontrivial* if its graph is not empty, i.e., if there exists at least an element $x \in X$ such that $F(x)$ is not empty.

3.2 The Viability Theorem

Let us consider a *control system* (U, f) defined by

- a feedback set-valued map $U : X \rightsquigarrow Z$
- a map $f : \mathrm{Graph}(U) \mapsto X$ describing the dynamics of the system

governing the evolution

$$\left\{ \begin{array}{lllll} i) & \text{for almost all } t, & x'(t) & = & f(x(t), u(t)) \\ ii) & \text{where} & u(t) & \in & U(x(t)) \end{array} \right. \tag{6}$$

Let us remark that when we take for controls the velocities, i.e., $U(x) := F(x)$ and $f(x, u) := u$, we find the usual differential inclusion $x' \in F(x)$. Conversely, the above system is the differential inclusion $x' \in F(x)$ in disguise where $F(x) := f(x, U(x))$.

We say that a closed subset $K \subset \mathrm{Dom}(U)$ is viable under (U, f) if from any initial state $x_0 \in K$ starts at least one solution on $[0, \infty[$ to the control system (6) *viable in* K (in the sense that for all $t \geq 0$, $x(t) \in K$).

The contingent cone was introduced by G. Bouligand in the early thirties: When K is a subset of X and x belongs to K, we recall that the *contingent cone* $T_K(x)$ to K at x is the closed cone of elements v such that

$$\liminf_{h \to 0+} \frac{d(x + hv, K)}{h} \; = \; 0$$

It happens to remain the best way to describe mathematical the concept of tangent direction to any subset.

We associate with any subset $K \subset \mathrm{Dom}(U)$ the regulation map $R_K : K \rightsquigarrow Z$ defined by

$$\forall \, x \in K, \;\; R_K(x) \; := \; \{u \in U(x) \mid f(x, u) \in T_K(x)\}$$

where $T_K(x)$ is the contingent cone to K at $x \in K$.

We say that K is a viability domain of (U, f) if and only if the regulation map R_K is strict (has nonempty values).

The Viability Theorem holds true for the class of Marchaud control systems, which satisfy the following conditions:

$$\left\{\begin{array}{ll} i) & \text{Graph}(U) \ \text{ is closed} \\ ii) & f \ \text{ is continuous} \\ iii) & \text{the velocity subsets } \ F(x) \ := \ f(x, U(x)) \ \text{ are convex} \\ iv) & f \ \text{ and } \ U \ \text{ have linear growth} \end{array}\right. \tag{7}$$

Theorem 3.1 (Viability Theorem) *Let us consider a Marchaud control system (U, f). Then a closed subset $K \subset \text{Dom}(U)$ is viable under (U, f) if and only if it is a* viability domain *of (U, f).*

Furthermore, any "open loop" *control $u(\cdot)$ regulating a viable solution $x(\cdot)$ in the sense that*

$$\text{for almost all } \ t, \ \ x'(t) = f(x(t), u(t))$$

obeys the regulation law

$$\text{for almost all } \ t, \ \ u(t) \in R_K(x(t)) \tag{8}$$

Otherwise, if K is not a viability domain of the control system (U, f), there exists a largest closed viability domain *of (U, f) contained in K (possibly empty), denoted* $\text{Viab}(K)$*, called the* viability kernel *of K, and equal to the set of states $x_0 \in K$ from which starts a solution of the control system viable in K.*

3.3 Toll Differential Inclusion

By using indicators, we can reformulate the differential inclusion

$$\text{for almost all } \ t, \ \ x'(t) \ \in \ F(x(t))$$

as

$$\text{for almost all } t, \ \ \psi_{F(x(t))}(x'(t)) \ < \ +\infty$$

Then we are led to define "fuzzy or toll dynamics" of a system by a toll set-valued map $\mathbf{F}$ associating with any $x \in X$ a toll set $\mathbf{F}(x)$ of velocities $\{v \mid F(x, v) < +\infty\}$. In this case, we can write the associated *fuzzy or toll differential inclusion* in the form

$$\text{for almost all } t \geq 0, \ \ F(x(t), x'(t)) < +\infty \tag{9}$$

or, equivalently, in the form

$$\text{for almost all } t \geq 0,\ x'(t) \in \mathbf{F}(x(t))$$

which is a toll subset instead of a usual subset. We associate with the toll dynamics its cheapest cost

$$\forall\, x \in \mathrm{Dom}(\mathbf{F}),\ \ \lambda_F(x) \ := \ \inf_{v \in X)} F(x,v)$$

When $\mu(x) \geq \lambda_F(x)$ is upper semicontinuous, we can associate the "level set-valued map" Λ_F^μ defined by

$$\Lambda_F^\mu(x) \ := \ \{v \in X \mid F(x,v) \leq \mu(x)\}$$

We observe that *the "level set-valued map" Λ_F^μ is a Marchaud map whenever* $\mathbf{F}$ *is a toll Marchaud map and μ is upper semicontinuous.*

The "cheapest set-valued map" $\Lambda_F := \Lambda_F^{\lambda_F}$ corresponds to the choice of $\mu = \lambda_F$. It is Marchaud whenever λ_F is upper semicontinuous. It is easy to check that *if the cost functions $F(\cdot, v)$ of the toll map* $\mathbf{F}$ *are continuous, then Λ_F is upper semicontinuous.*

Therefore, for any upper semicontinuous function $\mu \geq \lambda_F$, we say that the solutions of the differential inclusion

$$\text{for almost all } t \geq 0,\ \ x'(t) \ \in \ \Lambda_F^\mu(x(t))$$

are μ-selections of the toll differential inclusion. When λ_F is upper semicontinuous, the solutions to

$$\text{for almost all } t \geq 0,\ \ x'(t) \ \in \ \Lambda_F(x(t))$$

are called *cheapest solutions to the toll differential inclusion.*

3.4 Example: Random Differential Equations

Consider for instance random variables associating with each state of the system a random velocity. We consider the probability law by $d\mu(x)$ of the velocities, with which associate their average mean

$$\nabla C^\star_{\mu(x)}(0) \ = \ \int_{\mathbf{R}^n} y d\mu(x)(y)$$

where $C_{\mu(x)}$ denotes the Cramer transform of the state dependent probability law $d\mu(x)$ and $C^{\star}_{\mu(x)}$ its Fenchel transform, equal to the logarithm of its Laplace transform.

It is then natural to regard the differential equation

$$x'(t) \;=\; \int_{\mathbf{R}^n} y d\mu(x(t))(y)$$

as a random differential equation, associating with the state $x(t)$ at time t the state dependent average velocity.

By the Cramer transform, we know that this average velocity being the gradient at 0 of the conjugate function of the Cramer transform of $d\mu(x)$, it achieves the minimum of the cost function of the Cramer transform:

$$C_{\mu(x)}\left(\int_{\mathbf{R}^n} y d\mu(x)(y)\right) \;=\; \inf_{v\in X} C_{\mu(x)}(v)$$

Therefore, *the solution to the random differential equation is the cheapest solution of the toll differential inclusion*

$$\text{for almost all } t \geq 0,\;\; C_{\mu(x(t))}(x'(t)) \;<\; +\infty$$

3.5 Viability Theorem for Toll Differential Inclusions

We begin by characterizing usual subsets K enjoying the viability property for toll differential inclusion: from any initial state $x_0 \in K$ starts at least one solution $x(\cdot)$ to the toll differential inclusion (9) which is viable in K in the sense that

$$\forall\, t \geq 0,\;\; x(t) \;\in\; K$$

We shall denote by $\mathbf{R_K}$ the toll map associating with any $x \in K$ the toll intersection $\mathbf{R_K}(x) := \mathbf{F}(x) \cap T_K(x)$ of the toll set $\mathbf{F}(x)$ and the contingent cone $T_K(x)$. Its cost function is equal to

$$R_K(x,v) \;:=\; F(x,v) + \psi_{T_K(x)}(v)$$

Definition 3.2 *We shall say that* $\mathbf{R_K}$ *is the* toll regulation map (or the fuzzy controller) *associated with the viability constraints* K *and that a subset* $K \subset Dom(\mathbf{F})$ *is a viability domain of the toll set-valued map* $\mathbf{F}$ *if and only if*

$$\forall\, x \in K,\;\; \mathbf{R_K}(x) \;\neq\; \emptyset$$

i.e., if and only if

$$\forall\, x \in K,\;\; \exists\, v \;\text{ such that }\; R_K(x,v) \;<\; +\infty$$

We begin by proving an extension of the Viability Theorem (see Theorem 3.3.5 [13, Aubin]) to toll differential inclusions.

Theorem 3.3 (Toll Viability) *Let us consider a Marchaud toll set-valued map* $\mathbf{F}$ *from a finite dimensional vector-space* X *to itself. Any closed subset* $K \subset Dom(\mathbf{U})$ *enjoying the viability property with respect to* U *is a viability domain.*

The converse holds true if

$$\forall\, x \in \mathrm{Dom}(\mathbf{R_K}),\ \ \lambda_{R_K}(x) \ := \inf_{v \in T_K(x)} F(x,v) \ \leq\ \mu(x) \ <\ +\infty$$

where μ *is upper semicontinuous. In this case, from any initial state* $x_0 \in K$*, starts at least one viable* μ*-selection of the toll differential inclusion.*

When the toll set-valued map $\mathbf{F}$ is continuous, we can select a viable solution to the toll differential inclusion (9) which is *cheapest*, in the sense that the cost of its velocity's membership is minimal:

$$\text{for almost all } t,\ \ R_K(x(t), x'(t)) \ =\ \inf_{v \in T_K(x(t))} F(x(t), v) \qquad (10)$$

In order to obtain cheapest solutions, we have to check that the function λ_{R_K} is upper semicontinuous. We need naturally more regularity on the toll dynamics, but also on the closed subset K. We recall (see Chapter 5 of [13, Aubin] for instance) that K is *sleek* if the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous. Convex subsets are sleek, and for sleek subsets, the contingent cones are closed convex cones.

Theorem 3.4 *We posit the assumptions of Theorem 3.3. We assume moreover that the restriction of the membership function* U *to its domain (the graph of* $\mathbf{U}$*) is continuous and that the viability domain* K *is sleek.*

Then there exists a cheapest viable solution to the differential inclusion (9) (i.e., which satisfies condition (10)).

Proof — We introduce the function λ_{R_K} defined by

$$\lambda_{R_K}(x) \ :=\ \inf_{v \in T_K(x)} U(x,v) \ =\ \inf_{v \in X} R_K(x,v)$$

Since saying that K is sleek amounts to saying that the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous, the Maximum Theorem (see for instance Theorem 2.1.6 of [13, Aubin]) implies that the function λ is upper semicontinuous, because we have assumed that U is upper semicontinuous.

We then introduce the set-valued map G defined by

$$\Lambda_{R_K}(x) \ := \ \{ \ v \in T_K(x) \mid F(x,v) \leq \lambda_{R_K}(x) \ \}$$

Then Λ_{R_K} has a closed graph, and the other assumptions of the Viability Theorem are satisfied. There exists a viable solution to differential inclusion $x'(t) \in \Lambda_{R_K}(x(t))$, which is a cheapest viable solution to the toll differential inclusion (9.) $\square$

3.6 Example: Moreau Transform of a Differential Inclusion

Starting with a viability problem for a differential inclusion $x' \in F(x)$ when the subset K is not viable under F, one can try to regularize by the Moreau transform the subsets $F(x)$ to obtain toll subsets whose cost functions are

$$F_\sigma(x,v) \ = \ \frac{1}{\sigma^2} d(v, F(x))$$

The associated toll differential inclusion is defined by

$$\text{for almost all } \geq 0, \ \ d(x'(t), F(x(t))) \ < \ +\infty$$

and the cost function of the regulation map is equal to

$$R_K(x,v) \ := \ d(v, F(x)) + \psi_{T_K(x)}(v)$$

Therefore,

$$\lambda_{R_K}(x) \ = \ d(F(x), T_K(x))$$

Therefore, the cheapest solutions to this toll differential inclusion are the solutions to the projected differential inclusion

$$x'(t) \ \in \ \Pi_{T_K(x(t))}(F(x(t)))$$

It has solutions whenever the Marchaud map F is continuous and the subset K is sleek. This projected differential inclusion has been studied in Chapter 10 of [13, Aubin]. Their solutions are actually the same than the solutions to the variational inequalities

$$x'(t) \ \in \ F(x(t)) - N_K(x(t))$$

3.7 Toll Viability Domains

Is it possible to speak of toll subsets having the viability property?

A way to capture this idea is to introduce a continuous function[9] $\varphi : \mathbf{R}_+ \to \mathbf{R}$ with linear growth (which is used as a parameter in what follows) and the associated differential equation

$$w'(t) = -\varphi(w(t)), \;\; w(0) \; = \; V(x_0) \tag{11}$$

whose solutions $w(\cdot)$ set an upper bound to the membership of a toll subset when time elapses.

We shall say that a toll set $V \subset \mathrm{Dom}(\mathbf{F})$ enjoys the "toll viability property" (with respect to φ) if and only if from every initial state $x_0 \in \mathrm{Dom}(V)$ starts at least one solution to the toll differential inclusion (9) and to the differential equation (11) which are viable in the sense that

$$\forall\, t \geq 0, \;\; V(x(t)) \; \leq \; w(t), \;\; w(0) = V(x_0)$$

We have to adapt the concept of contingent cone to toll sets. This is possible by introducing the contingent cone to the epigraph of the cost function V of the toll set. We observe that the contingent cone to the epigraph of V at $(x, V(x))$ is the epigraph of a function denoted by $D_\uparrow V$:

$$\mathcal{E}p(D_\uparrow V(x)) \; = \; T_{\mathcal{E}p(V)}(x, V(x))$$

and which is equal to

$$\forall\, u \in X, \;\; D_\uparrow V(x)(u) \; = \; \liminf_{h \to 0+, u' \to u} (V(x + hu') - V(x))/h$$

We say that $D_\uparrow V(x)$ is the contingent epiderivative of V at $x \in \mathrm{Dom}(V)$.

We introduce now the "contingent set" $T_V^\varphi(x)$ (also denoted $T_V(x)$), the closed subset defined by:

$$T_V^\varphi(x) \; := \; \{\; v \in X \mid D_\uparrow V(x)(v) + \varphi(V(x)) \leq 0 \;\}$$

Definition 3.5 (Toll Viability Domain) *Let the continuous function φ with linear growth be given. We shall say that a toll subset V is a toll viability domain of a toll set-valued map $\mathbf{F}$ (with respect to φ) if and only if*

$$\forall\, x \in \mathrm{Dom}(V), \;\; \mathbf{R_V(x)} \; := \; \mathbf{F}(x) \cap T_V^\varphi(x) \; \neq \; \emptyset$$

[9] The main instance of such a function φ is the affine function $\varphi(w) := aw - b$, the solutions of which are $w(t) = (w(0) - \frac{b}{a})e^{-at} + \frac{b}{a}$.

i.e., if and only if

$$\forall\, x \in \mathrm{Dom}(V),\ \ \exists\, v \quad \textit{such that}\ \ R_K(x,v) := F(x,v) + T_V^{\varphi}(x) < +\infty$$

Let us set

$$\lambda_{R_V}(x) \ :=\ \inf_{v} R_K(x,v) \ =\ \inf_{v \in T_V^{\varphi}(x)} F(x,v)$$

Theorem 3.3 can be extended to toll viability domains:

Theorem 3.6 *The toll set-valued map* $\mathbf{F}$ *satisfies the assumptions of Theorem 3.3. We assume that* $V \subset \mathrm{Dom}(\mathbf{F})$ *is a closed toll subset which is contingently epidifferentiable*[10]. *If a closed toll subset* V *enjoys the viability property, then it is a closed toll viability domain of* $\mathbf{F}$ *and the converse holds true if*

$$\forall\, x \in \mathrm{Dom}(\mathbf{F}),\ \ \lambda_{R_V}(x) \ :=\ \inf_{v \in T_V^{\varphi}(x)} F(x,v) \ \leq\ \mu(x) \ <\ +\infty$$

where μ *is upper semicontinuous.*

We proceed by extending Theorem 3.4 on selection of toll viable solutions to toll differential inclusions which are cheapest, in the sense that

$$\text{for almost all } t,\ \ F(x(t), x'(t)) \ =\ \inf_{v \in T_V^{\varphi}(x(t))} F(x(t), v) \tag{12}$$

Theorem 3.7 *We posit the assumptions of Theorem 3.3. We assume moreover that the restriction of the membership function* F *to its domain (the graph of* $\mathbf{F}$*) is continuous and that the toll viability domain* V *satisfies*

$$x \rightsquigarrow T_V^{\varphi}(x) \ \text{ is lower semicontinuous}$$

Then there exists a cheapest viable solution to the differential inclusion (9) (which satisfies condition (12)).

Proof — The proof is the same as the one of Theorem 3.4, where the function μ is now defined by

$$\lambda_{R_V}(x) \ :=\ \inf_{v \in T_V^{\varphi}(x)} F(x,v) \ \ \Box$$

[10] This means that for all $x \in \mathrm{Dom}(V)$, $\forall\ v \in X$, $D_{\uparrow}V(x)(v) > -\infty$ and that $D_{\uparrow}V(x)(v) < \infty$ for at least a $v \in X$.

3.8 Toll Viability Kernels

Let us consider now any closed toll subset of the domain of $\mathbf{F}$, which is not necessarily a toll viability domain. The functions φ being given, we shall construct the largest closed toll viability domain V_φ contained in V.

Theorem 3.8 *The toll set-valued map satisfies the assumptions of Theorem 3.3. We assume that $V \subset \mathrm{Dom}(\mathbf{F})$ is a closed toll subset which is contingently epidifferentiable.*

Then for any upper semicontinuous function $\mu > 0$, there exists a largest closed toll viability domain V_φ^μ contained in V (for the toll differential inclusion), which enjoys furthermore the property:

$$\text{for almost all } t \geq 0,\;\; F(x(t), x'(t)) \;\leq\; \mu(x(t))$$

3.9 Fuzzy Control Problems

If $X,\ Z$ denote the state and control spaces of a control system

$$x'(t) \;=\; f(x(t), u(t)) \tag{13}$$

we consider a "fuzzy or toll" control as a toll set-valued map $\mathbf{U}$ defined by a cost function

$$(x, u) \in X \times Z \mapsto U(x, u) \in [0, +\infty] \tag{14}$$

We look for solutions to the toll control problem

$$\forall\, t \geq 0,\;\; u(t) \;\text{ belongs to the toll set }\; \mathbf{U}(x(t)) \tag{15}$$

in the sense that

$$\forall\, t \geq 0,\;\; U(x(t), u(t)) < +\infty \tag{16}$$

We observe that solutions to the toll control system are solutions to the toll differential inclusion $x' \in \mathbf{F}(x)$ where the toll set-valued map $\mathbf{F}$ is the product of $\mathbf{U}$ with the map $f(x, \cdot)$, the cost function of which is equal to

$$F(x, v) \;:=\; \inf_{\{u \,|\, f(x,u)=v\}} U(x, u)$$

Indeed, $F(x, v)$ is finite if and only if there exists a control $u \in \mathbf{U}(x)$ such that $f(x, u) = v$.

If we consider the toll viability problem of finding at least one viable solution

$$\forall\, t \geq 0,\;\; V(x(t)) \;\leq\; w(t) \tag{17}$$

of the toll control problem starting from every initial state $x_0 \in K$, we introduce the "toll regulation map" or "fuzzy controller" $\mathbf{R_V}$ defined by its cost function

$$R_V(x,u) \;=\; U(x,u) + \psi_{T_V^{\varphi}(x)}(f(x,u)) \tag{18}$$

In other words, $v \in \mathbf{R_V}(x)$ if and only if there exists a control $u \in \mathbf{U}(x)$ such that

$$D_{\uparrow}V(x)(f(x,u)) + \psi(V(x)) \;\leq\; 0$$

We set

$$\lambda_{R_V}(x) \;:=\; \inf_{v \in X} \inf_{\{u \,\mid\, f(x,u)=v\}} (U(x,u) + \psi_{T_V^{\varphi}(x)}(f(x,u)))$$

Hence we observe that the toll set V is a viability domain of the toll control system if and only if

$$\forall\, x \in \mathrm{Dom}(V),\;\; \mathbf{R_V}(x) \;\neq\; \emptyset$$

In this case, for any upper semicontinuous function $\mu \geq \lambda_{R_V}$, from any initial state $x_0 \in \mathrm{Dom}(V)$ starts at least one viable μ-selection of the toll control system, regulated by controls satisfying the toll regulation law

$$\text{for almost all } t \geq 0, \begin{cases} u(t) \in \mathbf{R_U}(x(t)) \\ U(x(t),u(t)) \;\leq\; \mu(x(t)) \end{cases}$$

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